Higher Mathematics

Ya. S. BUGROV S. M. NIKOLSKY

a collection of problems

MIR PUBLISHERS MOSCOW

ABOUT THE BOOK

This collection of about 1200 problems has been compiled for the following three textbooks by the same authors: *Fundamentals of Linear Algebra and Analytical Geometry, Differential and Integral Calculus, Differential Equations. Multiple Integrals. Series. Theory Functions of a Complex Variable*, thus completing a course in higher mathematics for engineering students and forming a four-book series entitled "Higher Mathematics". Academician S. Nikolsky is the author of the two-volume textbook *A Course of Mathematical Analysis* issued in English by Mir Publishers in 1977 and reprinted in 1981.

All the problems are provided with answers, some of the problems are supplied with hints. The book contains many worked problems.

At the beginning of each section references are given indicating the chapters and sections of the above mentioned books where the corresponding theoretical material can be found.

ИЗДАТЕЛЬСТВО
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Preface

This collection of problems has been compiled to suit the textbooks entering into our series "Higher Mathematics". These textbooks are referred to in the present collection as:

[1] – *Differential and Integral Calculus;*
[2] – *Fundamentals of Linear Algebra and Analytical Geometry;*
[3] – *Differential Equations. Multiple Integrals. Series. Theory of Functions of a Complex Variable*

Indicated at the beginning of each section of the present book are the chapter and section from the above mentioned textbooks where the relevant theoretical material can be found.

As a rule, each section contains a minimum number of problems which corresponds to the number of teaching periods assigned to study a certain topic. The odd-numbered problems may be recommended to be solved in auditorium, while the even-numbered problems must be solved by students at home independently.

The problems found in textbooks [1]-[3] may also be used for practical studies. These are not contained in the present Collection.

Yakov S. Bugrov
Sergei M. Nikolsky

Chapter 1

INTRODUCTION TO ANALYSIS

Sec. 1.1. Real Numbers. Sets

Using the method of mathematical induction, prove the following relationships:

1. $1+2+3+\ldots+n = n(n+1)/2$.

2. $1^2+2^2+\ldots+n^2 = n(n+1)(2n+1)/6$.

3. $(1+x)^n \geqslant 1+nx, \quad x > -1$.

4. $\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$.

To solve the below problems, it is necessary to study Chapter 1 from [1].

5. Let the set A consist of the youths of a given group, and the set B of the girls of the same group. Find $A \cup B$, $A \cap B$, $A \setminus B$. Also, consider the case when A or B is an empty set.

6. Let $A = \{2n\}$, $B = \{2n+1\}$. Find $A+B$, AB, $A \setminus B$ (n natural).

7. Which number is greater, a or b: $a = 1.(1234512)$, $b = 1.(12345)$; $a = 1.(12302)$, $b = 1.(123)$; $a = 1.(123412)$, $b = 1.(1234)$?

8. Find out to what number a the sequence of real numbers

$$a_1 = 0.1010101010\ldots,$$
$$a_2 = 0.1100110011\ldots,$$
$$a_3 = 0.111000111000\ldots,$$
$$a_4 = 0.111100001111\ldots,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_n = 0.\underbrace{11\ldots1}_{n \text{ times}}\underbrace{00\ldots0}_{n \text{ times}}1\ldots10\ldots0\ldots,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

is stabilized.

9. Find the sum of real numbers $a = 0.(12)$ and $b = 0.(13)$.

10. Given the sets $A = [2, 5]$ and $B = (3, 6)$. Find $A+B$, AB, $A \setminus B$.

11. Solve the following inequalities:

(a) $|x+3| < 0.1$; (b) $|x-3| \geqslant 10$;

(c) $|x| > |x+3|$; (d) $|3x-1| < |x-1|$;

(e) $\left|\frac{x-2}{3x+1}\right| \leqslant 1$.

12. Which of the two numbers is greater: a or $(-a)$?

13. Let $a \geqslant 0$. For what numbers b do the following relations take place

(a) $|a+b| = |a|+|b|$;

(b) $|a-b| = |a|+|b|$;

(c) $|a+b| < |a|+|b|$;

(d) $|a-b| < |a|+|b|$?

14. Find the modulus of the given number: (a) $\ln(1/e)$; (b) $\sin(3\pi/2)$, (c) $\cos(7\pi/4)$.

Sec. 1.2. Limit of a Sequence (See [1], Chapter 2)

15. Prove that

$$\lim_{n\to\infty} \frac{n+1}{n} = 1,$$

and determine for every $\varepsilon > 0$ the number $n_0 = n_0(\varepsilon)$ such that

$$\left|\frac{n+1}{n} - 1\right| < \varepsilon \quad \text{if} \quad n > n_0.$$

Fill in the table:

ε	0.1	0.001	0.00001	...
n_0				

In problems 16 to 19 find the indicated limits.

16. $\lim\limits_{n\to\infty} \frac{10^6 n}{n^2+1}$. **17.** $\lim\limits_{n\to\infty} \frac{n^\alpha \sin(n!)}{n+1}$ $(0 \leqslant \alpha < 1)$.

18. $\lim\limits_{n\to\infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \ldots + \frac{n-1}{n^2}\right)$.

19. $\lim\limits_{n\to\infty} \left(\frac{1^2}{n^3} + \frac{2^2}{n^3} + \ldots + \frac{(n-1)^2}{n^3}\right)$.

20. Prove that the variable α_n is an infinitesimal if

$$\alpha_n = \frac{n}{n^3+1}; \qquad \alpha_n = \frac{1}{n!}; \qquad \alpha_n = \frac{1}{\sqrt{n^2+1}}.$$

21. Prove that the variable β_n is an infinitely large quantity if

$$\beta_n = (-1)^n n^2; \qquad \beta_n = 2^{\sqrt{n}}; \qquad \beta_n = \ln(n+1).$$

22. Will the sequence $x_n = n^{(-1)^n/2}$ be infinitely large?

In Problems 23 and 24 prove the given equalities.

23. $\lim\limits_{n\to\infty} (\sqrt{3n+10} - \sqrt{3n}) = 0.$

24. $\lim\limits_{n\to\infty} \frac{3n^2+5}{2n^2+3n+1} = \frac{3}{2}.$

Using the theorem on existence of the limit of a monotone sequence, prove that the following sequences are convergent (Problems 25 and 26):

25. $x_n = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\ldots\left(1-\frac{1}{2^n}\right).$

26. $x_n = \frac{2}{1}\cdot\frac{3}{3}\cdot\frac{4}{5}\cdot\frac{5}{7}\cdots\frac{n+1}{2n-1}.$

27. Find the largest term of the following sequences:

$$x_n = \frac{n^2}{2^n}; \qquad x_n = \frac{\sqrt{n+1}}{10+n}.$$

28. Find the smallest term of the following sequences:

$$x_n = \left(1+\frac{1}{n}\right)^n; \qquad x_n = n^2-9n-10.$$

29. Find $\inf x_n$, $\sup x_n (n \in \mathbf{N})$, $\underline{\lim}\, x_n$, $\overline{\lim}\, x_n$ if

$$x_n = 1-\frac{1}{n}; \qquad x_n = \frac{(-1)^n}{n+1} + \frac{2+(-1)^n}{3} \quad (n \in \mathbf{N}).$$

30. Which numbers are the partial limits of the sequence

$$1, \frac{1}{2}, -1, 1, \frac{1}{3}, -1, 1, \frac{1}{4}, -1, 1, \ldots ?$$

A *partial limit* of an arbitrary bounded sequence is understood as the limit of its convergent subsequence. The existence of such subsequences in a bounded sequence is implied by the Bolzano-Weierstrass theorem.

Applying Cauchy's test for convergence, prove that the following sequences are convergent (Problems 31 to 33):

31. $x_n = \frac{\sin 1^2}{2} + \frac{\sin 2^2}{2^2} + \ldots + \frac{\sin n^2}{2^n} \equiv \sum_{k=1}^{n} \frac{\sin k^2}{2^k}$.

32. $x_n = \sum_{k=1}^{n} \frac{1}{k^2}$. 33. $x_n = \sum_{k=1}^{n} \frac{\cos k!}{k(k+1)}$.

Sec. 1.3. Functions. The Limit of a Function (See [1], Chapter 3)

Find the domain E of definition of the function $y = f(x)$ and the image $E_1 = f(E)$ of the set E with the aid of the function f (Problems 34 and 35).

34. $y = \frac{1}{1+x^2}$. 35. $y = \sqrt{2+3x-x^2}$.

36. Find $f(0)$, $f(x+2)$, $f(1/x)$, $f(x)+1$, $1/f(x)$ if

$$f(x) = \frac{1-x^2}{1+x^2}.$$

In Problems 37 to 41 find the graph of the given function.

37. $y = 8x-2x^2$. 38. $y = \frac{1-x^2}{1+x^2}$.

39. $y = -x^2+2x-1$. 40. $y = \frac{1-x}{1+x}$. 41. $y = \frac{3x+4}{4x-3}$.

42. Determine the lower and the upper bound to the set of values of the function $f(x)$ if

$$f(x) = x^2 \quad \text{on} \quad [-2, 5]; \quad \varphi(x) = x + \frac{1}{x} \quad \text{on} \quad (0, 3].$$

Hint. On the set $(0, 3]$ $\varphi(x) \geqslant 2$.

43. Construct the graphs of the functions

$$f(x) = \sup_{0 \leqslant t \leqslant x} \{\sin t\}; \quad \varphi(x) = \inf_{0 \leqslant t \leqslant x} \{\sin t\}.$$

In Problems 44 to 48 find the limits of the indicated functions.

44. (a) $\lim_{x \to 0} \frac{x^2-1}{2x^2-x-1}$; (b) $\lim_{x \to 1} \frac{x^2-1}{2x^2-x-1}$;

(c) $\lim_{x \to \infty} \frac{x^2-1}{2x^2-x-1}$; (d) $\lim_{x \to 0} \frac{(1+x)^3-(1+3x)}{x^2+3x^3}$;

(e) $\lim_{x \to 4} \frac{\sqrt{1+2x}-3}{\sqrt{x}-2}$; (f) $\lim_{x \to \infty} \left(\frac{x^2+3x-1}{2x^2-x+4}\right)^x$;

(g) $\lim\limits_{x\to 3}\dfrac{\sqrt{x+13}-2\sqrt{x+1}}{x^2-9}$; (h) $\lim\limits_{x\to -2}\dfrac{\sqrt[3]{x-6}+2}{x^3+8}$;

(i) $\lim\limits_{x\to a}\dfrac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}$;

(j) $\lim\limits_{x\to+\infty}[\sin\sqrt{x+1}-\sin\sqrt{x}]$.

45. (a) $\lim\limits_{x\to 0}\dfrac{\sin 3x}{x}$; (b) $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{x^2}$; (c) $\lim\limits_{x\to 0}\dfrac{\tan 4x}{x}$.

46. $\lim\limits_{x\to 0+0}\dfrac{\sin\sqrt{x}}{x}$.

47. (a) $\lim\limits_{x\to\infty}\left(1+\dfrac{2}{x}\right)^{2x}$; (b) $\lim\limits_{x\to 0}(1+3x)^{1/x}$;

(c) $\lim\limits_{x\to\pi/2}(\sin x)^{\tan x}$;

48. (a) $\lim\limits_{x\to 0}\dfrac{\ln(1+4x)}{x}$; (b) $\lim\limits_{x\to b}\dfrac{a^x-a^b}{x-b}$ $(a>0)$.

In Problems 49 to 56 investigate the given functions for continuity, represent the functions graphically, and determine the character of the points of discontinuity.

49. $f(x)=|x-1|$. **50.** $f(x)=\begin{cases}\dfrac{x^2-1}{x-1}, & x\neq 1,\\ A, & x=1.\end{cases}$

51. $f(x)=\operatorname{sgn}(x^2-2x-3)$. **52.** $y=\dfrac{1+x}{1+x^3}$.

53. $y=\dfrac{x}{1+x}$. **54.** $y=\operatorname{sgn}(\cos x)$.

55. $y=\begin{cases}2x, & 0\leqslant x\leqslant 1,\\ 2-x, & 1<x\leqslant 2.\end{cases}$ **56.** $f(x)=\begin{cases}\cos x, & x\leqslant 0,\\ a+x, & x>0.\end{cases}$

In Problems 57 and 58 prove that the functions $f(x)$ are uniformly continuous on $[a, b]$, i.e. $\forall\,\varepsilon>0$ there is $\delta(\varepsilon)>0$ (independent of the points of the interval) such that

$$|f(x_1)-f(x_2)|<\varepsilon,$$

whenever

$$|x_1-x_2|<\delta.$$

57. $f(x)=\begin{cases}x^2, & 0\leqslant x\leqslant 2,\\ 20-8x, & 2\leqslant x\leqslant 3.\end{cases}$

58. $f(x)=x^3$, $0\leqslant x\leqslant 2$.

59. Find the inverse of the function

$$y = \frac{ax+b}{cx+d} \quad (ad-bc \neq 0).$$

Let $x \to 0$. Separate the leading term of the form Ax^m (Problems 60 and 61).

60. $f(x) = 3x+x^4$. **61.** $f(x) = \sqrt{1+x}-\sqrt{1-x}$.

Sec. 1.4. Derivatives (See [1], Chapter 4)

62. Find $f'(0)$, $f'(2)$, if $f(x) = 2-2x+x^3$.

63. Find $f'(0)$, $f'(1)$, if $f(x) = x \arcsin \frac{x}{x+1}$.

In Problems 64 to 71 find the derivatives of the given functions.

64. $y = \frac{2x}{1-x^2}$.

65. (a) $y = x+\sqrt[3]{x}$; (b) $y = \frac{1}{x}+\frac{1}{\sqrt{x}}+\frac{1}{\sqrt[3]{x}}$;

(c) $y = \frac{x}{(1-x)^2(1+x)^3}$; (d) $y = x\sqrt{1+x^2}$;

(e) $y = \frac{x}{\sqrt{a^2-x^2}}$; (f) $y = \sqrt{x+\sqrt{x+\sqrt{x}}}$;

(g) $y = \sqrt[3]{x^2}-\frac{2}{\sqrt{x}}$; (h) $y = \tan\frac{x}{2}-\cot\frac{x}{2}$;

(i) $y = \frac{\sin x - x\cos x}{\cos x + x\sin x}$; (j) $y = \frac{1}{\cos^n x}$; (h) $y = 2^{\tan\frac{1}{x}}$;

(l) $y = e^{e^x}+e^{e^{e^x}}$; (m) $y = x^{a^a}+a^{x^a}$ $(a>0)$;

(n) $y = \arcsin\frac{1-x^2}{1+x^2}$; (o) $y = \arccos\sqrt{1-x^2}$.

66. $y = \tan x+\frac{1}{3}\tan^3 x$. **67.** $y = e^{-x^2}$.

68. (a) $y = \sin x^2$; (b) $y = \sin^2 x$; (c) $y = \sin^3 x^7$;
(d) $y = \cos(\sin x)$; (e) $y = \cos x^2$; (f) $y = \cos^2 x^4$.

69. (a) $y = \arcsin(x/a)$; (b) $y = \arctan(x/a)$;
(c) $y = \ln(x+\sqrt{a^2+x^2})$; (d) $y = \arcsin(\sin x)$;
(e) $y = \arccos(x/a)$, (f) $y = e^{x^3+x}$.

70. $y = \ln\tan(x/2)$.

71. (a) $y = x \arctan x$; (b) $y = \ln^3 x^2$;

(c) $y = \ln(\ln(\ln x))$; (d) $y = \frac{1}{4} \ln \frac{x^2-1}{x^2+1}$;

(e) $y = \sqrt{x+1} - \ln(1+\sqrt{x+1})$; (f) $y = \ln \tan\left(\frac{x}{2}+\frac{\pi}{4}\right)$;

(g) $y = \frac{1}{x}(\ln^3 x + 3\ln^2 x + 6\ln x + 6)$;

(h) $y = \frac{3}{2}(1-\sqrt[3]{1+x^2})^2 + 3\ln(1+\sqrt[3]{1+x^2})$;

(i) $y = \sqrt{x} - \arctan\sqrt{x}$; (j) $y = \arctan\frac{1+x}{1-x}$;

(k) $y = \frac{1}{12}\ln\frac{x^4-x^2+1}{(x^2+1)^2} - \frac{1}{2\sqrt{3}}\arctan\frac{\sqrt{3}}{2x^2-1}$;

(l) $y = \frac{x^6}{1+x^{12}} + \arctan x^6$; (m) $y = \arctan(\tan^2 x)$;

(n) $y = x\arctan x - 0.5\ln(1+x^2) - 0.5(\arctan x)^2$;

(o) $y = \arctan(x+\sqrt{1+x^2})$.

(p) There takes place the formula

$$\frac{d}{dx}\begin{vmatrix} a_{11}(x) & \dots & a_{1n}(x) \\ \dots & \dots & \dots \\ a_{n1}(x) & \dots & a_{nn}(x) \end{vmatrix} = \sum_{k=1}^{n} \begin{vmatrix} a_{11}(x) & \dots & a_{1n}(x) \\ \dots & \dots & \dots \\ a_{k-1,1}(x) & \dots & a_{k-1,n}(x) \\ a'_{k1}(x) & \dots & a'_{kn}(x) \\ a_{k+1,1}(x) & \dots & a_{k+1,n}(x) \\ \dots & \dots & \dots \\ a_{n1}(x) & \dots & a_{nn}(x) \end{vmatrix},$$

where the elements of the determinant $a_{ij}(x)$ are differentiable functions. Hence, the derivative of a determinant of order n is equal to the sum of n determinants of order n each of which differs from the original determinant by that the appropriate row in it is replaced by the row made up from the derivatives of the elements of the replaced row.

Prove the differentiation formula for determinants of the second and third order.

In Problems 72 and 73 find the derivatives and construct the graphs of the given functions and their derivatives.

72. $y = \begin{cases} 1-x, & -2 < x < 1, \\ (1-x)(2-x), & 1 \leqslant x \leqslant 2, \\ -(2-x), & 2 < x < 4. \end{cases}$

73. $y = \begin{cases} x, & x < 0, \\ \ln(1+x), & x \geqslant 0. \end{cases}$

Find the logarithmic derivatives (i.e. y'/y) of the indicated functions y (Problems 74 and 75).

74. $y = x\sqrt{\dfrac{1-x}{1+x}}$.

75. $y = \cosh^2 x$.

In Problems 76 to 79 find the derivatives of the indicated hyperbolic functions.

76. (a) $y = \sinh(x^2+1)$; (b) $y = \sinh^3 x^6$.

77. $y = \cosh^2(x^2+x+1)$.

78. (a) $y = \tanh^2 x$; (b) $y = \tanh x^2$.

79. (a) $y = \tanh(\ln x+1)$; (b) $y = \operatorname{Arsinh} x$;

(c) $y = \operatorname{Arsinh}(x+\sqrt{1+x^2})$; (d) $y = \ln\sinh x$;

(e) $y = \cosh\ln x$; (f) $y = e^{\tanh x}$;

(g) $y = (\sinh x)^{\cosh x}$ $(x > 0)$; (h) $y = \dfrac{1}{\cosh^3 x}$;

(i) $y = \tanh\dfrac{x}{2} - \coth\dfrac{x}{2}$.

80. For the function $f(x) = x^2+x+1$ determine the differential and the increment at the point $x = 1$ for $\Delta x = 0.1$.

In Problems 81 to 84 find the differentials of the given functions.

81. $d(xe^x)$.

82. $d(\sinh x)$.

83. $d(\sinh x - x\cosh x)$.

84. $d(\ln(1-x^2))$.

85. Find the second-order derivative of the following functions: (a) $y = e^{-x^2} \equiv \exp(-x^2)$; (b) $y = x\sqrt{1+x^2}$.

86. Let there be given the Wronskian

$$W(x) = \begin{vmatrix} y_1(x) & \dots & y_n(x) \\ y_1'(x) & \dots & y_n'(x) \\ \dots\dots\dots & \dots & \dots\dots\dots \\ y_1^{(n-1)}(x) & \dots & y_n^{(n-1)}(x) \end{vmatrix},$$

where the functions $y_1(x), \dots, y_n(x)$ are continuous on (a, b) together with their derivatives up to order n inclusive.

Prove that

$$\frac{dW}{dx} = \begin{vmatrix} y_1(x) & \dots & y_n(x) \\ y_1'(x) & \dots & y_n'(x) \\ \dots\dots\dots & \dots & \dots\dots\dots \\ y_1^{(n-2)}(x) & \dots & y_n^{(n-2)}(x) \\ y_1^{(n)}(x) & \dots & y_n^{(n)}(x) \end{vmatrix}.$$

87. $y = x^5$, find d^4y.

88. $y = e^x \ln x$, find d^3y.

In Problems 89 to 92 find the derivatives y_x' and y_{x^2}'' of the functions represented parametrically if:

89. $x = 2t - t^2$, $y = 3t - t^3$. **90.** $x = 2\cos t$, $y = 2\sin t$.

91. $x = f'(t)$, $y = tf'(t) - f(t)$.

92. $x = t - \sin t$, $y = 1 - \cos t$.

93. (a) Write the equations of the tangent and the normal to the curve $y = 2 + x - x^3$ at the point $A = (2, -4)$.

(b) Find out whether or not the graphs of the functions $y = \sinh x$ and $y = \ln(1+2x)$ have a common tangent at the point (0, 0).

The *angle between two curves* $y = f_1(x)$ and $y = f_2(x)$ at the point of their intersection with the abscissa $x = x_0$ is defined as the angle φ between the tangents to the curves at that point. Therefore

$$\tan\varphi = \tan(\varphi_2 - \varphi_1) \frac{\tan\varphi_2 - \tan\varphi_1}{1 + \tan\varphi_1 \tan\varphi_2} = \frac{f_2'(x_0) - f_1'(x_0)}{1 + f_1'(x_0) f_2'(x_0)},$$

where φ_1, φ_2 are the angles formed by the indicated tangents with the x-axis (see Fig. 1).

94. At what angle do the curves $y = \sin x$ and $y = \cos x$ $(0 < x < \pi)$ intersect?

95. At what angle do the curves $y = x^\alpha$ and $y = x^{\frac{1+\alpha}{1-\alpha}}$ $(0 < \alpha < 1)$ intersect at the point (1, 1)?

96. At what angle does the curve $y = \ln\left(1 + (x/\sqrt{3})\right)$ intersect the x-axis?

97. (a) Check the validity of Rolle's theorem for the function $y = (x-1)(x-2)$ on [1, 2].

(b) The polynomial $P_4(x) = x^4 - x^3 + x^2 - x$ has the root $x = 1$. Prove that the polynomial $\frac{d}{dx} P_4(x)$ has a real root belonging to the open interval (0, 1).

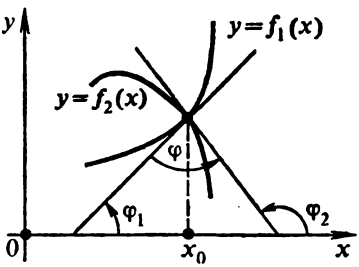

Fig. 1.

(c) Prove that all the roots of the polynomial $P_n(x) = \frac{d^n}{dx^n}(1-x^2)^n$ are real and belong to the interval $(-1, 1)$.

98. Check the validity of Lagrange's theorem $f(b)-f(a) = f'(c)(b-a)$ for the function

(a) $y = 1+x+x^3$;
(b) $f(x) = x^3+Ax^2+Bx+C$, where A, B, C are real numbers;
(c) $f(x) = Ax^3+Bx+C$; on $[0, 1]$. Find the point c.

99. Prove that if a function $f(x)$ has a bounded derivative on (a, b) ($|f'(x)| \leqslant M$), then

(a) $f(x)$ is uniformly continuous on (a, b).
(b) If a and b are finite numbers, then $f(x)$ is bounded on (a, b).

Hint. Let x be an arbitrary point, and let x_0 be a fixed point within (a, b). Then

$$|f(x)| = |f(x)-f(x_0)+f(x_0)|$$
$$\leqslant |f(x)-f(x_0)|+|f(x_0)| = |f'(c)|\,|x-x_0|+|f(x_0)|,$$

where c is found between the points x and x_0.

(c) If the interval (a, b) is infinite, then the function $f(x)$ may be unbounded. Consider the function $f(x) = \ln x$ on $(1, \infty)$.

100. Determine the intervals of monotonicity for the functions

(a) $y = 3+x-x^2$; (b) $y = 4x-x^4$.

If a function $\varphi(x)$ is continuous on $[a, b]$ and has a positive (negative) derivative on (a, b), then it is increasing (decreasing) on $[a, b]$.

This fact can be utilized when proving inequalities.

For instance, the function $\varphi(x) = e^x-1-x$ is continuous on $[0, \infty)$. It is increasing on $[0, \infty)$, since $\varphi'(x) = e^x-1 > 0$ on $(0, \infty)$. Further, $\varphi(0) = 0$, therefore

$$e^x-1-x > 0, \quad \forall x \in (0, \infty).$$

On $(-\infty, 0)$ the function $\varphi(x)$ is decreasing, therefore $e^x-1-x > \varphi(0) = 0$. Hence, $\forall x \neq 0$

$$e^x > 1+x.$$

In Problems 101 and 102 prove the indicated inequalities.

101. $x-\frac{x^3}{6} < \sin x < x \quad (x > 0)$.

102. $\cos x > 1-\frac{x^2}{2} \quad (x > 0)$.

In Problems 103 to 109 compute the indicated limits using L'Hospital's rule.

103. $\lim\limits_{x\to 0}\dfrac{\sin ax}{\sin bx}$. **104.** $\lim\limits_{x\to 0}\dfrac{\tan 2x}{x}$.

105. $\lim\limits_{x\to 0}\dfrac{\arcsin 2x-2\arcsin x}{x^3}$. **106.** $\lim\limits_{x\to 0}\dfrac{\tan 2x-\sin 2x}{x^3}$.

107. $\lim\limits_{x\to\infty}\dfrac{x^2+2x}{4x^2+1}$. **108.** $\lim\limits_{x\to+0}x^x$.

109. (a) $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{1/x^2}$; (b) $\lim\limits_{x\to 0}\left(\dfrac{\tan x}{x}\right)^{1/x^2}$;

(c) $\lim\limits_{x\to 0}\dfrac{\sinh ax}{\sinh bx}$; (d) $\lim\limits_{x\to 0}\dfrac{\tanh 3x}{\tan x}$;

(e) $\lim\limits_{x\to 0}\dfrac{x\cot x-1}{x^2}$; (f) $\lim\limits_{x\to 0}\dfrac{\tanh x-x}{x-\sinh x}$;

(g) $\lim\limits_{x\to\pi/2}\dfrac{\tan 3x}{\tan x}$; (h) $\lim\limits_{x\to 0}\left(\coth x-\dfrac{1}{x}\right)$;

(i) $\lim\limits_{x\to 1}x^{\frac{1}{1-x}}$; (j) $\lim\limits_{x\to+0}\left(\ln\dfrac{1}{x}\right)^x$; (k) $\lim\limits_{x\to+\infty}(\tanh x)^x$;

(l) $\lim\limits_{x\to 0}\left(\dfrac{\cos x}{\cosh x}\right)^{1/x^2}$; (m) $\lim\limits_{x\to a}\dfrac{a^x-x^a}{x-a}$ $(a>0)$;

(n) $\lim\limits_{x\to 0}\dfrac{\sqrt[n]{1+x}-1}{x}$ (n—an integer);

(o) $\lim\limits_{x\to 0}\dfrac{\sqrt[m]{1+\alpha x}-\sqrt[n]{1+\beta x}}{x}$ (m, n—integers);

(p) $\lim\limits_{x\to 1}\dfrac{\sqrt[m]{x}-1}{\sqrt[n]{x}-1}$;

(q) $\lim\limits_{x\to\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right)$. *Hint.* Multiply and divide by the conjugate expression;

(r) $\lim\limits_{x\to 1}(1-x)\tan\dfrac{\pi x}{2}$; (s) $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{\sin^3 x}$.

110. Is it possible to apply L'Hospital's rule in

$$\lim_{x\to\infty}\frac{x-\sin x}{x+\sin x}?$$

In Problems 111 to 114 write the expansion of the given functions in powers of x.

111. $f(x) = \tan x$ up to the term x^5.

112. $f(x) = e^{2x-x^2}$ up to the term x^3.

113. $y = \ln \cos x$ up to the term x^4.

114. $y = \sin(\sin x)$ up to the term x^3.

In Problems 115 to 118 find the indicated limits using the expansion by Taylor's formula.

115. $\lim\limits_{x\to 0} \dfrac{\cos x - \exp(-x^2/2)}{x^4}$.

116. $\lim\limits_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{\sin x}\right)$.

117. $\lim\limits_{x\to 0} \dfrac{x-\sinh x}{x^3}$.

118. $\lim\limits_{x\to 0} \dfrac{1-\cosh x^2}{x^4}$.

119. Applying Taylor's formula compute the following numbers:

(a) e with an accuracy to 10^{-6};
(b) $\sin 1°$ with an accuracy to 10^{-6};
(c) $\sqrt{5}$ with an accuracy to 10^{-3};
(d) $\ln 2$ and $\ln 3$ with an accuracy to 10^{-5} (see [1], Sec. 9.14, (8)).

120. Investigate the following functions for a local extremum:

(a) $y = 2-x-x^2$; (b) $y = |x|$;

(c) $y = 2x^2-x^4$; (d) $y = x+\dfrac{1}{x}$;

(e) $y = e^x \sin x \quad (0 \leqslant x \leqslant 2\pi)$; (f) $y = 1/(4+x^2)$;

(g) $y = x/(1+4x^2)$; (h) $y = x^3-6x^2+9x-4$;

(i) $y = \cos x + 0.5 \cos 2x$;

(j) $y = \sin x + 0.5 \sin 2x$.

121. Find the greatest and the least values of the given function:
(a) $y = 2^x$ on $[0, 5]$; (b) $y = x+\dfrac{1}{x}$ on $\left[\dfrac{1}{2}, 10\right]$.

122. Find the distance from the curve $y = x^2$ to the straight line $y-x+2 = 0$.

123. Find sup and inf of the following functions:

(a) $y = \begin{cases} 1, & 0 \leqslant x \leqslant 1, \\ 4-2x, & 1 < x \leqslant 2; \end{cases}$ (b) $y = \dfrac{1+x^2}{1+x^4}$ on $(0, \infty)$

124. Find the intervals of concavity and points of inflection for the following functions: (a) $y = 3x^2-x^3$; (b) $y = \exp(-x^2)$.

125. Find the asymptotes to the graphs of the following functions:

(a) $y = x + \frac{1}{x}$; (b) $y = e^{1/x}$;

(c) $y = \exp(-x^2)$; (d) $y = \ln(1+e^x)$.

126. Graph the below functions on having carried out a complete investigation of their behaviour (extrema, points of inflection, zeros of function, direction of concavity, asymptotes):

(a) $y = \frac{x-3}{\sqrt{1+x^2}}$; (b) $y = \frac{e^x}{1+x}$.

127. Graph the functions represented parametrically (see Sec. 4.22 [1]).

(a) $x = 2t - t^2$, $y = 3t - t^3$; (b) $x = \frac{2+t^2}{1+t^2}$, $y = \frac{t^3}{1+t^2}$.

128. (a) In an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ inscribe a rectangle with sides parallel to the coordinate axes having the greatest area (see Fig. 2).

(b) Given a perfectly square sheet of tin with side a. Small squares of equal size with side x are cut away from its corners to make a rectangular box by bending the sheet along the dashed lines (Fig. 3). For what x will the volume of the box be the greatest?

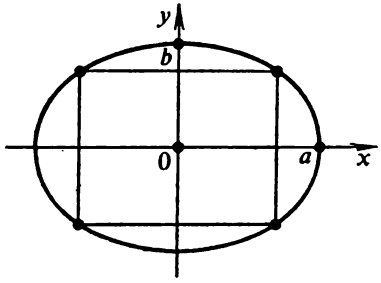

Fig. 2.

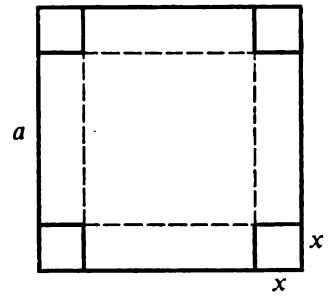

Fig. 3.

(c) The ship K (see Fig. 4) is situated at a distance of 9 km from the nearest point B of the rectilinear seashore. A messenger has to be sent from the ship to the camp L found on the shore at a distance of 15 km from the point B (as measured along the shore). At what point P must the messenger pull in to the shore to get to the camp in the shortest time if he goes on foot at a speed of 5 km/hr and by boat at 4 km/hr?

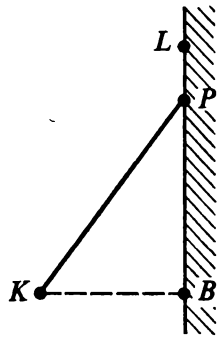

Fig. 4.

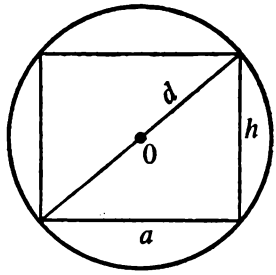

Fig. 5.

(d) From a round log of diameter d a girder of a rectangular cross-section with base a and altitude h has to be made up (Fig. 5). For what a and h will the girder be the strongest if it is known that the strength of a girder is proportional to ah^2?

129. In the parabola given by the equation $y = 3-x^2$ inscribe a rectangle having the greatest area so that one of its sides lie on the x-axis and two of its vertices on the parabola (Fig. 6).

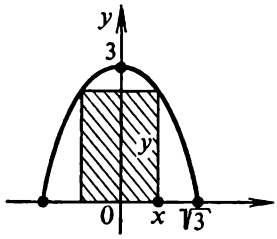

Fig. 6.

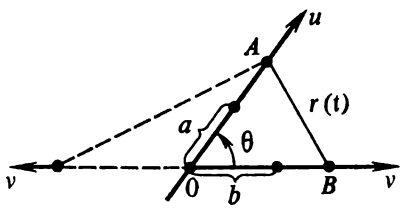

Fig. 7.

130. Two ships A and B sail at constant speeds u and v along straight lines forming an angle θ. Determine the shortest distance between the ships if at a certain instant of time their distances from the point of intersection of their paths were equal to a and b, respectively (Fig. 7).

131. Determine the radius of curvature of the given curves:

(a) $y^2 = 2px$; (b) $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$;

(c) $y = \frac{1}{2}(1-\cos t)$, $x = \frac{1}{2}(t-\sin t)$ (cycloid).

132. Write the equation of the evolute of the cycloid

$$x = a(t-\sin t), \quad y = a(1-\cos t).$$

Chapter 2

INTEGRALS

Sec. 2.1. Indefinite Integral (See [1], Chapter 5)

Using the table integrals, compute the following integrals (Problems 133 to 143):

133. $\int x^2(5-x)^4\,dx$. **134.** $\int (1-x^2)^2\,dx$. **135.** $\int \frac{x^2\,dx}{1+x^2}$.

136. $\int (1+\sin x+\cos x)\,dx$. **137.** $\int \frac{x^2+1}{x^2-1}\,dx$.

138. $\int \tan^2 x\,dx$. **139.** $\int \tanh^2 x\,dx$. **140.** $\int \coth^2 x\,dx$.

141. $\int \frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1-x^4}}\,dx$. **142.** $\int (2x-3)^{99}\,dx$.

143. (a) $\int \frac{dx}{2+3x^2}$; (b) $\int (1-x)(1-2x)(1-3x)\,dx$;

(c) $\int \left(\frac{a}{x}+\frac{a^2}{x^2}+\frac{a^3}{x^3}\right)dx$; (d) $\int \frac{\sqrt{x}-2\sqrt[3]{x^2}+1}{\sqrt[4]{x}}\,dx$;

(e) $\int \left(\frac{1-x}{x}\right)^3 dx$; (f) $\int \frac{x+2}{\sqrt{x}}\,dx$; (g) $\int \frac{x^3}{1+x^4}\,dx$;

(h) $\int \frac{x^2}{1-x^2}\,dx$; (i) $\int \frac{x^2+3}{x^2-1}\,dx$; (j) $\int (2^x+3^x)^2\,dx$;

(k) $\int (a\sinh x+b\cosh x)\,dx$; (l) $\int (2x-9)^{10}\,dx$;

(m) $\int \frac{dx}{\sqrt{2-3x^2}}$; (n) $\int \frac{dx}{\sqrt{2+3x^2}}$;

(o) $\int \frac{dx}{\sqrt{3x^2-2}}$; (p) $\int (e^{-2x}+e^{-3x})\,dx$.

Transforming the integrand in a proper way or using suitable substitutions, compute the following integrals (Problems 144 to 152):

144. $\int \frac{x\,dx}{\sqrt{1-x^2}}$. **145.** $\int \frac{e^x\,dx}{2+e^x}$. **146.** $\int \frac{dx}{\cos^2 x\sin^2 x}$.

147. $\int \frac{dx}{\sqrt{1+x}+\sqrt{x-1}}$. **148.** $\int \tan x\,dx$. **149.** $\int \frac{dx}{(1+x)\sqrt{x}}$.

150. $\int x\sqrt{2-5x}\,dx$. **151.** $\int \frac{2^x 3^x\,dx}{9^x-4^x}$.

152. (a) $\int \frac{x^2\,dx}{\sqrt{x^2-2}}$; (b) $\int x^2\sqrt[3]{1+x^3}\,dx$;

(c) $\int \frac{x\,dx}{(1+x^2)^2}$; (d) $\int \frac{x^3\,dx}{x^8-2}$; (e) $\int \frac{dx}{e^x+e^{-x}}$;

(f) $\int \frac{dx}{\sqrt{1+e^{2x}}}$; (g) $\int \frac{\ln^2 x}{x}\,dx$; (h) $\int \cot x\,dx$;

(i) $\int \frac{dx}{\sinh x}$; (j) $\int \frac{dx}{\cosh x}$; (k) $\int \frac{\arctan x}{1+x^2}\,dx$;

(l) $\int \sinh^2 x\,dx$; (m) $\int \cosh^2 x\,dx$; (n) $\int \frac{dx}{\sinh^2 x\cosh^2 x}$;

(o) $\int \frac{x\,dx}{\sqrt[3]{1-3x}}$; (p) $\int x^3\sqrt[3]{1+x^2}\,dx$; (q) $\int \frac{dx}{1+e^x}$.

Applying the method of integration by parts, compute the following integrals (Problems 153 to 160):

153. $\int \arctan x\,dx$. **154.** $\int xe^{-x}\,dx$. **155.** $\int x\sinh x\,dx$.

156. $\int x\sin x\,dx$. **157.** $\int \arcsin x\,dx$.

158. $\int x\cos x\,dx$. **159.** $\int \arctan\sqrt{x}\,dx$.

160. (a) $\int x^2\sin 2x\,dx$; (b) $\int x\arctan x\,dx$;

(c) $\int x^2e^{-2x}\,dx$; (d) $\int x^3e^{-x^2}\,dx$; (e) $\int \ln x\,dx$;

(f) $\int x^n\ln x\,dx$ $(n\neq -1)$; (g) $\int x^3\cosh 3x\,dx$;

(h) $\int x^2\sinh x\,dx$; (i) $\int \sin x\ln(\tan x)\,dx$.

Applying the method of undetermined coefficients, find the integrals of the indicated rational functions (Problems 161 to 164):

161. (a) $\int \frac{2x+3}{(x-2)(x+5)}\,dx$; (b) $\int \frac{2x+1}{(x-2)^3(x+5)}\,dx$.

162. (a) $\int \frac{dx}{x^2-2x+2}$; (b) $\int \frac{dx}{x^2-2x+1}$; (c) $\int \frac{2x-2}{x^2-2x+2}\,dx$.

163. (a) $\int \frac{dx}{(x+1)(x^2+1)}$; (b) $\int \frac{dx}{(x^2+1)^2(x+2)}$.

164. (a) $\int \frac{dx}{(x+1)(1-x)^2}$; (b) $\int \frac{x\,dx}{(x+1)(x+2)(x+3)}$;

(c) $\int \frac{x^{10}\,dx}{x^2+x-2}$; (d) $\int \frac{dx}{x^3+1}$; (e) $\int \frac{x\,dx}{x^3-1}$;

(f) $\int \frac{dx}{x^4-1}$; (g) $\int \frac{dx}{x^4+1}$.

In Problems 165 and 166 integrate the given linear-fractional surds:

165. $\int \frac{dx}{1+\sqrt{x}}$.

166. (a) $\int \frac{dx}{(1+\sqrt[4]{x})^2\sqrt{x}}$; (b) $\int \frac{dx}{x(1+2\sqrt{x}+\sqrt[3]{x})}$;

(c) $\int \frac{dx}{\sqrt{x}(1+\sqrt[4]{x})^3}$; (d) $\int \frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}\,dx$;

(e) $\int \frac{dx}{1+\sqrt{x}+\sqrt{1+x}}$;

(f) $\int \frac{dx}{\sqrt[n]{(x-a)^{n+1}\cdot(x-b)^{n-1}}}$ (n—a natural number).

In Problems 167 and 168 compute the integrals of the indicated quadratic surds.

167. $\int \frac{dx}{\sqrt{1+x+x^2}}$. **168.** $\int \sqrt{x^2-2x+2}\,dx$.

In Problems 169 to 172 integrate the indicated trigonometric functions:

169. (a) $\int \sin^4 x\,dx$ (b) $\int \frac{dx}{\sin 2x}$.

170. (a) $\int \cos^3 x\,dx$; (b) $\int \frac{dx}{\cos 2x}$.

171. $\int \frac{dx}{2\sin x-\cos x-1}\left(t=\tan\frac{x}{2}\right)$.

172. (a) $\int \frac{dx}{a^2\sin^2 x+b^2\cos^2 x}$ ($t=\tan x$);

(b) $\int \frac{dx}{1+\cos x}$; (c) $\int \frac{dx}{1-\cos x}$; (d) $\frac{dx}{1+\sin x}$;

(e) $\int [\tanh(2x+1)+\coth(2x-1)]\,dx$.

Sec. 2.2. Definite Integral (See [1], Chapter 6)

173. Prove that the Dirichlet function

$$y = \begin{cases} 0, & x\text{—irrational}, \\ 1, & x\text{—rational}, \end{cases}$$

is not integrable on any closed interval $[a, b]$.

174. Without evaluating the below integrals, find out which of them is greater:

(a) $\int_0^{\pi/2} \sin x\,dx, \quad \int_0^{\pi/2} x\,dx, \quad \int_0^{\pi/2} \frac{2x}{\pi}\,dx;$

(b) $\int_0^1 e^x\,dx, \quad \int_0^1 (1+x)\,dx.$

In Problems 175 to 179 evaluate the indicated definite integrals with the aid of the Newton-Leibniz formula:

175. (a) $\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{dx}{1+x^2}$; (b) $\int_0^{\pi/4} \sin 2x\,dx$;

(c) $\int_0^{\pi/2} \cos x\,dx$; (d) $\int_0^{\pi/4} \tan x\,dx$; (e) $\int_{-1/2}^{1/2} \frac{dx}{\sqrt{1-x^2}}$;

(f) $\int_{\sinh 1}^{\sinh 2} \frac{dx}{\sqrt{1+x^2}}$; (g) $\int_0^2 |1-x|\,dx$; (h) $\int_1^2 x \ln x\,dx.$

176. $\int_0^{\pi/2} \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} \quad (a, b > 0).$

177. $\int_0^{\pi} x \sin x\,dx.$ **178.** $\int_0^{\ln 2} xe^{-x}\,dx.$

179. $\int_0^1 xf''(x)\,dx.$

180. Prove that for integral k and l:

(a) $\int_0^{2\pi} \sin kx \sin lx\,dx = \begin{cases} 0, & \text{if } k \neq l, \\ \pi, & \text{if } k = l; \end{cases}$

(b) $\int_0^{2\pi} \cos kx \cos lx\, dx = \begin{cases} 0, & \text{if } k \neq l, \\ \pi, & \text{if } k = l; \end{cases}$

(c) $\int_0^{2\pi} \cos kx \sin lx\, dx = 0, \quad \forall k, l.$

Hint. Transform the integrand to obtain a sum of trigonometric functions.

181. Find the derivative of the given integral:

(a) $\frac{d}{dx}\int_a^x \sin t^2\, dt$; (b) $\frac{d}{da}\int_a^b \sqrt{1+t^2}\, dt$; (c) $\frac{d}{dx}\int_a^b \sin x^2\, dx$.

182. Check to see whether the mean-value theorem

$$\int_a^b f(x)\, dx = f(\xi)(b-a)$$

is fulfilled for the function $f(x) = x^2$ on $[0, 1]$.

183. Evaluate the definite integral of the function

$$f(x) = \begin{cases} 1, & 0 \leqslant x \leqslant 1, \\ 4-2x, & 1 < x \leqslant 2. \end{cases}$$

Sec. 2.3. Applications of Definite Integral (See [1], Chapter 7)

In Problems 184 to 187 compute the area of the figure bounded by the indicated curves.

184. $y = x^2$, $y = 2-x$.

185. $y = h\left(1-\frac{4x^2}{b^2}\right)$, $y = 0$ $(h > 0, b > 0)$ (Fig. 8).

186. $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$.

187. $r = a(1+\cos\varphi)$, r, φ are polar coordinates (see Fig 9).

188. Compute the arc length of the curve

$$y = \frac{x^2}{4}-\frac{1}{2}\ln x \quad (1 \leqslant x \leqslant e).$$

189. Find the arc length of the astroid (Fig. 10)

$$x^{2/3}+y^{2/3} = a^{2/3}.$$

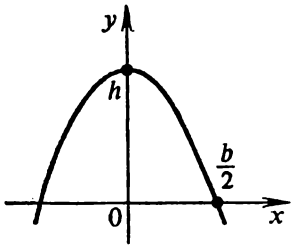

Fig. 8.

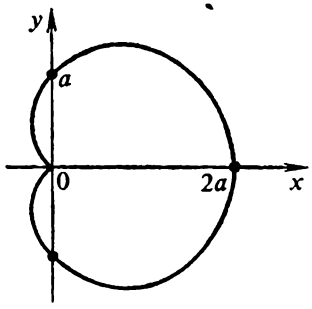

Fig. 9.

190. Find the arc length of one arch of the cycloid (Fig. 11)

$$x = a(t - \sin t), \quad y = a(1 - \cos t) \quad (0 \leqslant t \leqslant 2\pi).$$

191. Derive the formula for the arc length of a curve given in polar coordinates by the equation $\varrho = f(\theta)$.

192. Find the arc length of the curve given by the equation $r = a\cos^3 \frac{\varphi}{3}$ in polar coordinates (Fig. 12).

193. Find the arc length of the cardioid $r = a(1+\cos\varphi)$ (see Fig. 9).

194. Find the volume of the

(a) cone with the generatrix $y = \frac{r}{h}x \quad (0 \leqslant x \leqslant h)$ (Fig. 13);

(b) conical barrel with the radii of its bases r_1 and r_2 and altitude h (Fig. 14);

(c) solid generated by revolving one arch of the sinusoid $y = \sin x \quad (0 \leqslant x \leqslant \pi)$;

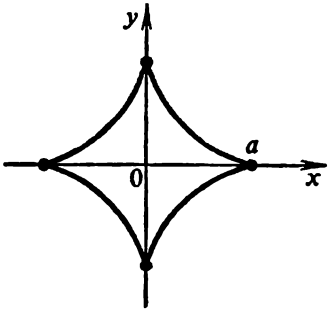

Fig. 10.

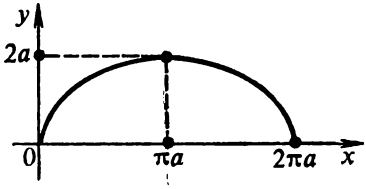

Fig. 11.

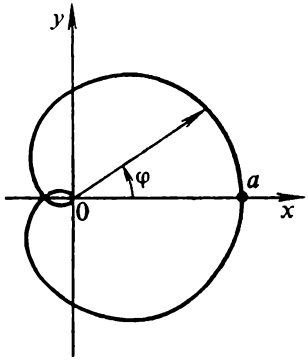

Fig. 12.

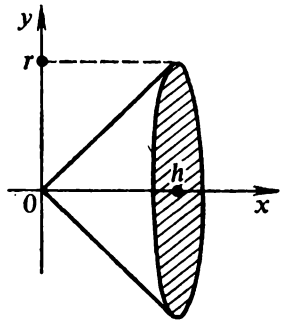

Fig. 13.

(d) solid generated by rotating the parabola $y^2 = 2px$ about the x-axis $(0 \leqslant x \leqslant a)$ (Fig. 15).

195. Compute the surface area formed by revolving the curve

(a) $x = a(t - \sin t)$, $y = a(1 - \cos t)$ $(0 \leqslant t \leqslant 2\pi)$ about the axis of symmetry of the curve;

(b) $9y^2 = x(3-x)^2$ $(0 \leqslant x \leqslant 3)$ about the x-axis;

(c) $y = x^3$ $(0 \leqslant x \leqslant 1)$ about the x-axis;

(d) $x = a\cos^3 t$, $y = a\sin^3 t$ (astroid) about the x-axis.

196. Evaluate the given integrals by the rectangle formula, trapezoid rule, Simpson's rule and estimate the error:

(a) $\int_0^1 \frac{dx}{1+x}$ $(n = 8)$; (b) $\int_0^1 \frac{dx}{1+x^3}$ $(n = 12)$.

197. Construct Lagrange's interpolation polynomial of the third degree for the function $f(x)$ if $f(0) = 0$, $f(1) = 1$, $f(2) = 2$, $f(3) = 0$.

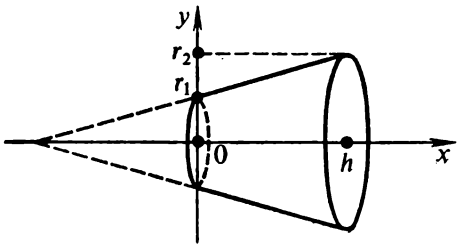

Fig. 14.

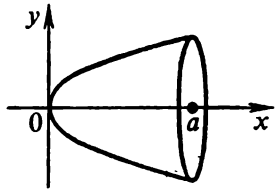

Fig. 15.

Sec. 2.4. Improper Integrals (See [1], Chapter 6)

198. Evaluate the indicated integrals:

(a) $\int_0^{\infty} \frac{x\,dx}{1+x^4}$; (b) $\int_0^{1} \frac{dx}{(1-x)^{\alpha}}$ $(\alpha > 0)$;

(c) $\int_0^{1} \frac{dx}{\sqrt{1-x^2}}$; (d) $\int_0^{1/2} \frac{dx}{x \ln x}$.

199. Investigate the given integrals for convergence applying the comparison test:

(a) $\int_0^{\infty} \exp(-x^2)\,dx$; (b) $\int_1^{\infty} \frac{dx}{\sqrt{1+x^4}}$;

(c) $\int_1^{\infty} \frac{\sin x}{x^2}\,dx$; (d) $\int_0^{1} \frac{dx}{\sqrt[3]{1-x^4}}$.

200. Compute the below integrals using the method of integration by parts:

(a) $\int_0^{\infty} xe^{-x}\,dx$; (b) $\int_0^{\infty} e^{-ax}\cos bx\,dx$ $(a > 0)$;

(c) $\int_0^{\pi} e^x \sin x\,dx$.

201. Investigate the given integrals for convergence:

(a) $\int_0^{\infty} \frac{\cos ax\,dx}{1+x^p}$ $(p \geqslant 0)$; (b) $\int_0^{\infty} \frac{\sin x\,dx}{x^p+x^q}$ $(p, q \geqslant 0)$;

(c) $\int_0^{\infty} \frac{x^m\,dx}{1+x^n}$ $(n \geqslant 0)$.

Chapter 3

FUNDAMENTALS OF LINEAR ALGEBRA AND ANALYTICAL GEOMETRY

Sec. 3.1. Determinants and Matrices (see [2], Secs. 1-3)

In Problems 202 to 210 evaluate the given determinants:

202. $\begin{vmatrix} 5 & 3 \\ 6 & 4 \end{vmatrix}$. **203.** $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$. **204.** $\begin{vmatrix} 8 & 5 \\ 3 & 2 \end{vmatrix}$.

205. $\begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix}$. **206.** $\begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\beta & \cos\beta \end{vmatrix}$.

207. $\begin{vmatrix} a+b & a-b \\ a-b & a+b \end{vmatrix}$. **208.** $\begin{vmatrix} \frac{x}{1+x} & \frac{2x+1}{1+x} \\ \frac{-1}{1+x} & \frac{x}{1+x} \end{vmatrix}$.

209. $\begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$. **210.** $\begin{vmatrix} 0 & x & 0 \\ x & 1 & x \\ 0 & x & 0 \end{vmatrix}$.

211. Find out whether the given permutation is even or odd:

(a) 1, 2, 4, 3, 5; (b) 5, 1, 2, 3, 4;

(c) 1, 3, 2, 5, 4; (d) 1, 4, 3, 2, 5.

212. Find the cofactors of all the elements in the determinant

$$\Delta = \begin{vmatrix} a & x & x \\ x & b & x \\ x & x & c \end{vmatrix}$$

and check to see that

$$\Delta = \sum_{k=1}^{3} a_{1k} A_{1k}.$$

213. Evaluate the below determinants by accumulating zeros in a row or a column.

(a) $\begin{vmatrix} 1 & 1 & 4 & 1 \\ 2 & 1 & 3 & 0 \\ 3 & 1 & 2 & 1 \\ 4 & 1 & 1 & 0 \end{vmatrix}$; (b) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 4 \\ 3 & 3 & 3 & 4 \\ 4 & 4 & 4 & 4 \end{vmatrix}$.

214. Evaluate the following Vandermonde determinants:

(a) $\begin{vmatrix} 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 5 & 5^2 & 5^3 \\ 1 & 6 & 6^2 & 6^3 \end{vmatrix}$; (b) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$.

215. Multiply the determinants

$$\Delta_1 = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 2 \\ 2 & 3 & 2 \end{vmatrix} = 5 \quad \text{and} \quad \Delta_2 = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 7$$

using all the four possible ways (i. e. by multiplying the rows or columns of Δ_1 by the rows or columns of Δ_2). Check to be sure that in all the cases the product of the determinants $\Delta = \Delta_1\Delta_2$ is equal to 35.

216. Let there be given two matrices

$$A = \begin{pmatrix} 3 & 4 \\ 5 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 8 & 1 \\ 2 & 3 \end{pmatrix}.$$

Find the matrix $C = \lambda A + \mu B$ if (a) $\lambda = 1$, $\mu = 2$; (b) $\lambda = -5$, $\mu = 2$.

217. For the matrix

$$A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 2 & 2 & 1 \end{pmatrix}$$

find its transpose, or the conjugate matrix, A^* and determine the rank of A.

Sec. 3.2. Systems of Linear Equations (See [2], Sec. 4)

In Problems 218 to 222 solve the given systems of equations using Cramer's rule.

218. $\begin{cases} 2x_1 + 3x_2 = 1, \\ 3x_1 + 5x_2 = 4. \end{cases}$ **219.** $\begin{cases} 3x_1 + x_2 = 4, \\ 2x_1 + 4x_2 = 1. \end{cases}$

220. $\begin{cases} x + y = 1, \\ x - y = 2. \end{cases}$ **221.** $\begin{cases} 2x - y + 3z = 9, \\ 3x - 5y + z = -4, \\ 4x - 7y + z = 5. \end{cases}$

222. $\begin{cases} x- y+3z = 9, \\ 3x-5y+ z=-4, \\ 4x-7y+ z= 5. \end{cases}$

223. By transforming the augmented matrix B find out whether the system

$$\begin{cases} 2x+7y+3z+ t = 5, \\ x+ 3y+5z-2t = 3, \\ x+ 5y-9z+8t = 1, \\ 5x+18y+4z+5t = 12 \end{cases}$$

is solvable.

224. Find the ranks of the matrices

$$A = \begin{pmatrix} 2 & 1 & 4 & 5 \\ 1 & 0 & 1 & 2 \\ 1 & 2 & 4 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 3 & 1 & 3 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

by transforming the rows and columns of the given matrices (by accumulating zeros).

Sec. 3.3. Vectors (See [2], Sec. 5)

225. (a) Find the projection of the vector $\mathbf{a} = (1, 4)$ on the direction determined by the vector $\mathbf{b} = (1/\sqrt{2}, 1/\sqrt{2})$.

(b) Compute the projections x, y, z of the vector $\mathbf{a}$ on the coordinate axes if $|\mathbf{a}| = 2$, $\alpha = \pi/4$, $\beta = \pi/3$, $\gamma = 2\pi/3$, where α, β, γ are the angles formed by the vector $\mathbf{a}$ with the x-, y-, and z-axes, respectively.

(c) Find the projections of the vector $\mathbf{a}$ from (b) on the directed straight line L with the unit vector $\mathbf{b} = (1/2, 1/2, 1/\sqrt{2})$.

226. Let there be given two vectors: $\mathbf{a} = (1, 2, 2)$, $\mathbf{b} = (2, 1, -1)$. Find the moduli of these vectors, the distance between the points $\mathbf{a}$ and $\mathbf{b}$ (provided that the vectors $\mathbf{a}$ and $\mathbf{b}$ are laid off from the origin) and the scalar product $\mathbf{ab}$.

227. Find the cosine of the angle between the following pairs of vectors:

(a) $\mathbf{a} = (2, -4, 4)$, $\mathbf{b} = (-3, 2, 6)$;

(b) $\mathbf{a} = (\sqrt{2}, 1, -1)$, $\mathbf{b} = (1, 0, 0)$;

(c) $\mathbf{a} = (1, 3, \sqrt{6})$, $\mathbf{b} = (1, 1, 0)$.

228. Can the vector $\mathbf{a} = (x, y, z)$ make the angles $\alpha = \pi/6$, $\beta = \pi/4$ with the coordinate axes?

229. Find the coordinates of vector $\mathbf{a}$ if $|\mathbf{a}| = 3$, $\alpha = \beta = \gamma$.

230. Given two vectors $\mathbf{a}$ and $\mathbf{b}$ such that $|\mathbf{a}| = 13$, $|\mathbf{b}| = 19$, $|\mathbf{a}+\mathbf{b}| = 24$. Find $|\mathbf{a}-\mathbf{b}|$.

231. Given two vectors $\mathbf{a}$ and $\mathbf{b}$ such that $|\mathbf{a}| = 11$, $|\mathbf{b}| = 23$, $|\mathbf{a}-\mathbf{b}| = 30$. Find $|\mathbf{a}+\mathbf{b}|$.

232. Find the angle between the vectors $\mathbf{a} = (1, 1, 1, 1)$ and $\mathbf{b} = (0, 1, 0, 1)$.

233. Let the vectors $\mathbf{a}$ and $\mathbf{b} \neq 0$ be orthogonal. For what value of the parameter λ is the vector $\mathbf{a}+\lambda\mathbf{b}$ orthogonal to the vector $\mathbf{a}+\mathbf{b}$?

Sec. 3.4. Dividing a Line Segment in a Given Ratio (See [2], Sec. 7)

234. On the line segment in the three-dimensional space R_3, connecting the points $O = (0, 0, 0)$ and $A = (1, 2, 2)$, find the point $M = (x, y, z)$ dividing this segment in a ratio $2:3$.

235. Find the coordinates of the centre of gravity $M = (x, y)$ of the system of two material points $A = (3, -5)$, $B = (-1, 1)$ at which the masses $q = p = 1$ are concentrated.

236. If in Problem 235 $q = 3$, $p = 5$, find the coordinates of the centre of gravity.

237. A line segment with the end points $A = (1, -5)$, $B = (4, 3)$ is divided into three equal parts. Determine the coordinates of the division points.

Sec. 3.5. The Straight Line (See [2], Sec. 8)

238. Find out which of the points $M_1 = (3, 1)$, $M_2 = (2, 3)$, $M_3 = (-2, 1)$ lie on the straight line $2x+3y-13 = 0$.

239. Write the equation of the straight line $2x+3y-13 = 0$ in the slope form and also as the equation of a straight line passing through some point in a given direction.

240. Given a straight line $x+2y+1 = 0$. Write the equation of the straight line passing through the point $M_0 = (2, 1)$: (a) parallel to the given line; (b) perpendicular to the given line.

241. Given: the equations of two sides of a rectangle

$$2x-3y+5 = 0, \quad 3x+2y-7 = 0$$

and one of its vertices $O = (0, 0)$. Write the equations of two other sides of this rectangle.

242. Reduce the equations of straight lines: (a) $2x+5y+4=0$; (b) $x+y-1=0$; (c) $2x-y+3=0$ to the normal form.

243. Find the distance from the point $A=(1, 2)$ to the straight line: (a) $2x+4y-5=0$; (b) $2x+8y+1=0$; (c) $x+y=0$.

Sec. 3.6. The Plane (See [2], Sec. 9)

244. Write the equation of the plane passing through the point $M_0=(1, 2, -3)$ perpendicular to the vector $\mathbf{v}=(1, -2, 3)$.

245. Write the equation of the plane passing through the three points: $(1, 1, 1)$, $(1, -1, 0)$, $(2, 1, 3)$.

246. Write the equation of the plane passing through the point $(1, 1, 1)$: (a) perpendicular and (b) parallel to the plane

$$2x+4y+z-5=0.$$

247. Which pairs of the below equations define parallel planes

(a) $4x+2y-4z+5=0$, $2x+y-2z-1=0$;

(b) $x-3z+2=0$, $2x-6z-7=0$;

(c) $2x-3y+5z-7=0$, $4x-6y+10z-14=0$;

(d) $2x-3y+5z-7=0$, $4x-3y+10z-14=0$.

248. Reduce the equation of the given plane: (a) $2x-3y+6z-7=0$; (b) $4x-y+8z-14=0$ to the normal form.

249. Find the distance from the point $A=(1, 2, 1)$ to the indicated plane: (a) $2x-3y+6z-7=0$; (b) $2x+y-2z-1=0$.

250. Find the angle between the planes from the preceding problem.

251. Write the equation of the spherical surface with centre at the origin touching the plane $2x+3y+4z-12=0$.

252. Write the equation of the plane $2x+y-5z-6=0$ in the intercept form.

253. Write the equation of the plane which passes through the point $(2, -1, 1)$ perpendicular to the planes $2x-y+3z-1=0$ and $x+2y+z=0$.

254. Determine the angles α, β, γ formed by the normal to the plane: (a) $x+y\sqrt{2}+z-1=0$; (b) $x\sqrt{3}+y+1=0$ with the coordinate axes.

255. Compute the distance between the parallel planes:

(a) $x-2y-2z-1=0$, $x-2y-2z-6=0$;

(b) $2x-3y+6z-1=0$, $4x-6y+12z+1=0$.

Sec. 3.7. A Straight Line in Space (See [2], Sec. 10)

256. Find the points at which the straight line

$$\begin{cases} 2x+y-z-3=0, \\ x+y+z-1=0 \end{cases}$$

pierces the coordinate planes.

257. Find the relations which must be satisfied by the coefficients of the straight line

$$\begin{cases} A_1x+B_1y+C_1z+D_1=0, \\ A_2x+B_2y+C_2z+D_2=0, \end{cases}$$

in order that the latter: (a) intersect the axis of abscissas, (b) coincide with it.

258. Write the canonical equations of the straight line passing through the point $(1, 0, -1)$ parallel to the vector $\mathbf{a} = (2, -3, 5)$.

259. Write the parametric equations of the straight line passing through the point $(1, -1, -3)$ parallel to the vector $\mathbf{a} = (2, 1, 5)$.

260. Write the canonical equations of the straight line:

(a) $\begin{cases} x-2y+3z-4=0, \\ 3x+2y-5z-4=0; \end{cases}$ (b) $\begin{cases} 5x+y+z=0, \\ 2x+3y-2z+5=0. \end{cases}$

261. Find the angle φ between the straight lines

$$\frac{x-2}{1}=\frac{y-3}{-1}=\frac{z}{\sqrt{2}}, \quad \frac{x+1}{1}=\frac{y-2}{1}=\frac{z+5}{\sqrt{2}}.$$

262. Given two straight lines

$$\frac{x+2}{2}=\frac{y}{-3}=\frac{z-1}{4}, \quad \frac{x-3}{l}=\frac{y-1}{4}=\frac{z-7}{2}.$$

For what value of l do they intersect?

263. Write the equation of the straight line passing through the point $(1, -1, 0)$ perpendicular to the plane $2x-4y+z=3$.

Sec. 3.8. Orientation of a System of Vectors. The Vector Product of Two Vectors. Triple Scalar Product (See [2], Secs. 10-13)

264. Given two vectors:

(a) $\mathbf{a} = (1, 2)$, $\mathbf{b} = (3, 5)$; (b) $\mathbf{a} = (1, 2)$, $\mathbf{b} = (3, 7)$.

Find out their orientation with respect to the xy-system.

265. Let the vectors $\mathbf{a}$ and $\mathbf{b}$ form an angle $\varphi = \pi/6$ and let $|\mathbf{a}| = 7$, $|\mathbf{b}| = 6$. Find $|\mathbf{a} \times \mathbf{b}|$.

266. Find out whether the vectors $\mathbf{a} = (1, 0, 3)$ and $\mathbf{b} = (2, 0, 6)$ are collinear.

267. What condition must be satisfied by the vectors $\mathbf{a}$ and $\mathbf{b}$ in order that the vectors $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$ be collinear?

268. Prove that if $\mathbf{a}+\mathbf{b}+\mathbf{c} = 0$, then $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}$.

269. Compute the sine of the angle formed by the vectors $\mathbf{a} = (-2, 2, 1)$ and $\mathbf{b} = (6, 3, 2)$.

270. Find the area S of the parallelogram constructed on the plane vectors $\mathbf{a} = (1, 2)$ and $\mathbf{b} = (3, 4)$.

Remark 1. If a triangle with vertices $A = (x_1, y_1)$, $B = (x_2, y_2)$, $C = (x_3, y_3)$ is given in the xy-plane, then the area of this triangle is obviously equal to half the area of the parallelogram constructed on the vectors $\overrightarrow{AB} = (x_2 - x_1, y_2 - y_1)$ and $\overrightarrow{AC} = (x_3 - x_1, y_3 - y_1)$. Hence, the area of the triangle ABC is equal to

$$S = \left| \frac{1}{2} \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix} \right|.$$

This equality can also be written in the form

$$S = \left| \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|.$$

This determinant of the third order is equal to the second-order determinant written above. To make sure that it is so, it suffices to multiply the first row of the third-order determinant by (-1) and to add to the second and third rows and then to expand the determinant in terms of the elements of the third row. As an example, compute the area of the triangle ABC with vertices $A = (1, 2)$, $B = (2, -1)$, $C = (0, 1)$.

271. Check to see whether the below vectors are coplanar:

(a) $\mathbf{a} = (2, 3, -1)$, $\mathbf{b} = (1, -1, 3)$, $\mathbf{c} = (1, 9, -11)$;

(b) $\mathbf{a} = (1, 1, 0)$, $\mathbf{b} = (0, 1, 0)$, $\mathbf{c} = (1, 1, 1)$.

272. Prove that the four points $A = (1, 2, -1)$, $B = (0, 1, 5)$, $C = (-1, 2, 1)$, $D = (2, 1, 3)$ lie in one plane.

Remark 2. The Mutual Positions of Two Straight Lines.

Let us be given two straight lines

$$\frac{x - x_1}{\alpha_1} = \frac{y - y_1}{\beta_1} = \frac{z - z_1}{\gamma_1}, \quad (L_1)$$

$$\frac{x - x_2}{\alpha_2} = \frac{y - y_2}{\beta_2} = \frac{z - z_2}{\gamma_2}, \quad (L_2)$$

where $\alpha_i^2+\beta_i^2+\gamma_i^2 = 1$ $(i = 1, 2)$. We introduce (unit) vectors

$$\mathbf{a}^1 = (\alpha_1, \beta_1, \gamma_1), \quad \mathbf{a}^2 = (\alpha_2, \beta_2, \gamma_2),$$

and points

$$A_1 = (x_1, y_1, z_1), \quad A_2 = (x_2, y_2, z_2).$$

The distance d between two straight lines L_1 and L_2 is defined as the minimum of distances between two arbitrary points $A \in L_1$ and $B \in L_2$.

Three cases are possible here:

I. L_1 and L_2 intersect at some point. In this case, obviously, $d = 0$.

II. L_1 and L_2 are *skew* lines, i.e. they do not intersect and are not parallel to each other. In this case the vectors $\mathbf{a}^1$ and $\mathbf{a}^2$ are noncollinear and the distance d between L_1 and L_2 is computed by the formula

$$d = \left|\overrightarrow{A_1A_2}\right| \frac{(\mathbf{a}^1\times\mathbf{a}^2)}{|\mathbf{a}^1\times\mathbf{a}^2|} = \frac{1}{|\mathbf{a}^1\times\mathbf{a}^2|} \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{vmatrix}. \quad (1)$$

Indeed, let Π_1 be the plane passing through L_1 parallel to L_2 and let Π_2, be the plane passing through L_2 parallel to L_1. It is obvious that the planes Π_1 and Π_2 are parallel to each other and perpendicular to the vector $\mathbf{a}^1\times\mathbf{a}^2$. Therefore, the distance between the straight lines L_1 and L_2 is equal to the distance between the planes Π_1 and Π_2. Since the point $A_1 \in L_1 \in \Pi_1$, and the point $A_2 \in L_2 \in \Pi_2$, the distance d between Π_1 and Π_2 is, obviously equal to the absolute value of the projection of the vector $\overrightarrow{A_1A_2}$ on the vector $\mathbf{a}^1\times\mathbf{a}^2$:

$$d = \left|\mathrm{pr}_{\mathbf{a}^1\times\mathbf{a}^2}\overrightarrow{A_1A_2}\right|,$$

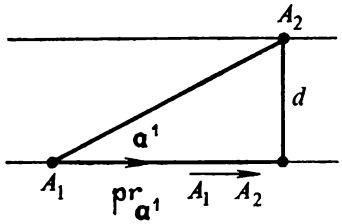

Fig. 16.

which equals the right-hand member of (1) (see [2], Sec. 5, p. 39).

III. L_1 and L_2 are parallel. In this case we may regard that $\mathbf{a}^1 = \mathbf{a}^2$ (changing, if necessary, the sign in the equations of L_2). The distance d between L_1 and L_2 is computed by the formula

$$d = \sqrt{\left|\overrightarrow{A_1A_2}\right|^2 - \left(\mathrm{pr}_{\mathbf{a}^1}\overrightarrow{A_1A_2}\right)^2} = \sqrt{\left|\overrightarrow{A_1A_2}\right|^2 - (\overrightarrow{A_1A_2}, \mathbf{a}^1)^2}$$

$$= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2-[(x_2-x_1)\alpha_1+(y_2-y_1)\beta_1+(z_2-z_1)\gamma_1]^2}$$

$$(\alpha_1^2+\beta_1^2+\gamma_1^2 = 1).$$

(see Fig. 16).

Remark 3. Two Straight Lines Belonging to One Plane.
Let us show that for two straight lines L_1 and L_2 to belong to some plane, it is necessary and sufficient that the equality

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{vmatrix} = 0 \qquad (2)$$

be fulfilled.

Indeed, this equality can be written in vector form in the following way: $\overrightarrow{A_1A_2}(\mathbf{a}^1 \times \mathbf{a}^2) = 0$. But this is just the condition of belonging of the three vectors $\overrightarrow{A_1A_2}$, $\mathbf{a}^1$, $\mathbf{a}^2$ to one plane (see [2], Sec. 13) and, hence, the condition of belonging of the straight lines L_1 and L_2 to one plane.

Remark 4. In Cases I and III the lines L_1 and L_2 are, obviously, found in one plane. Therefore, for them equality (2) is fulfilled. As far as Case II is concerned, it is obvious that L_1 and L_2 do not belong to any one plane and, therefore, in this case equality (2) is not fulfilled.

Given below are some exercises in finding the distance between the indicated pairs of straight lines:

(a) $\frac{x-1}{4} = \frac{y-2}{5} = \frac{z-6}{4}$, $\frac{x-1}{5} = \frac{y-2}{6} = \frac{z-6}{3}$;

(b) $\frac{x-1}{4} = \frac{y-2}{5} = \frac{z-6}{4}$, $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{1}$;

(c) $\frac{x-1}{4} = \frac{y-2}{5} = \frac{z-1}{3}$, $\frac{x-5}{8} = \frac{y-3}{10} = \frac{z+1}{6}$;

(d) $\frac{x-2}{1} = \frac{y+1}{2} = \frac{z}{3}$, $\frac{x-2}{2} = \frac{y+2}{4} = \frac{z+1}{6}$;

(e) $\frac{x}{4} = \frac{y-1}{5} = \frac{z+2}{4}$, $\frac{x-1}{1} = \frac{y+2}{2} = \frac{z-6}{1}$;

(f) $\frac{x+1}{1} = \frac{y+1}{2} = \frac{z-1}{1}$, $\frac{x+1}{2} = \frac{y+1}{3} = \frac{z-1}{1}$;

(answers: (a) 0; (b) $3/\sqrt{2}$; (c) $\sqrt{33/2}$; (d) $\sqrt{3/14}$; (e) $7/\sqrt{2}$; (f) 0).

Let us consider the solution of Problem (b). In this example $A_1 = (1, 2, 6)$, $A_2 = (0, 1, 2)$. The vectors (4, 5, 4) and (1, 2, 1) are not unit vectors. Multiplying the equations of the given lines by the moduli of these vectors, we obtain the equations of the lines in the desired form:

$$\frac{x-1}{4/\sqrt{57}} = \frac{y-2}{5/\sqrt{57}} = \frac{z-6}{4/\sqrt{57}}, \quad \frac{x}{1/\sqrt{6}} = \frac{y-1}{2/\sqrt{6}} = \frac{z-2}{1/\sqrt{6}},$$

i.e. $\alpha_1 = 4/\sqrt{57}$, $\beta_1 = 5/\sqrt{57}$, $\gamma_1 = 4/\sqrt{57}$; $\alpha_2 = 1/\sqrt{6}$, $\beta_2 = 2/\sqrt{6}$, $\gamma_2 = 1/\sqrt{6}$. It is easy to check that condition (2) is not fulfilled, i.e. our lines are skew lines. Therefore, the required distance will be found by formula (1). Let us find the vector product of the unit vectors $\mathbf{a}^1 = (\alpha_1, \beta_1, \gamma_1)$, $\mathbf{a}^2 = (\alpha_2, \beta_2, \gamma_2)$:

$$\mathbf{a}^1 \times \mathbf{a}^2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{vmatrix} = \frac{1}{\sqrt{6 \cdot 57}} [-3\mathbf{i} + 3\mathbf{k}].$$

Hence, $|\mathbf{a}^1 \times \mathbf{a}^2| = 1/\sqrt{19}$. Now by formula (1), we obtain

$$d = \left| \sqrt{19} \begin{vmatrix} -1 & -1 & -4 \\ 4 & 5 & 4 \\ 1 & 2 & 1 \end{vmatrix} \frac{1}{\sqrt{57 \cdot 6}} \right| = \frac{3}{\sqrt{2}}.$$

Remark 5. The Volume of Tetrahedron.

Let in space there be given a tetrahedron (triangular pyramid) $ABCD$ with vertices $A = (x_1, y_1, z_1)$, $B = (x_2, y_2, z_2)$, $C = (x_3, y_3, z_3)$, $D = (x_4, y_4, z_4)$. It is required to find the volume of this tetrahedron (Fig. 17).

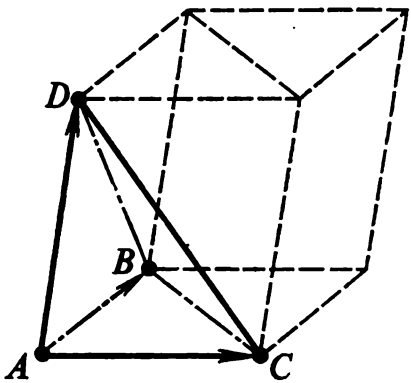

Fig. 17.

As is seen from the figure, the volume of the tetrahedron $ABCD$ is equal to 1/6 the volume of the parallelepiped constructed on the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$. But we know (see [2], Sec. 13) that the volume of this parallelepiped is equal to the absolute value of the triple scalar product of the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$. Therefore, the volume V of the triangular pyramid $ABCD$ is equal to

$$V = \left| \frac{1}{6} (\overrightarrow{AB} \times \overrightarrow{AC}) \overrightarrow{AD} \right|$$

$$= \left| \frac{1}{6} \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} \right| = \left| \frac{1}{6} \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \right|.$$

Problems. Find the volume of the tetrahedron given by its vertices:

(a) $A = (0, 0, 0)$, $B = (1, 1, 0)$, $C = (2, 1, 0)$, $D = (0, 0, 6)$;

(b) $A = (0, 0, 0)$, $B = (4, 1, 1)$, $C = (1, 1, 0)$, $D = (0, 0, 8)$;

(answers: (a) 1; (b) 4).

Sec. 3.9. Dependent and Independent Systems of Vectors (See [2], Sec. 14)

273. Find out whether the vectors

(a) $\mathbf{a}^1 = (1, 1, 1, 1)$, $\mathbf{a}^2 = (1, 2, 1, 2)$,
$\mathbf{a}^3 = (3, 1, 3, 1)$, $\mathbf{a}^4 = (0, 1, 1, 0)$;

(b) $\mathbf{a}^1 = (1, 0, 1)$, $\mathbf{a}^2 = (1, 1, 2)$, $\mathbf{a}^3 = (2, 1, 2)$

are linearly dependent or linearly independent.

274. Prove that a system of vectors containing two equal vectors are linearly dependent.

275. Find all the values of λ for which the vector $\mathbf{b} = (7, -2, \lambda)$ is linearly expressed in terms of the vectors $\mathbf{a}^1 = (2, 3, 5)$, $\mathbf{a}^2 = (3, 7, 8)$, $\mathbf{a}^3 = (1, -6, 1)$.

Sec. 3.10. Linear Operators. The Basis (See [2], Secs. 15-17)

276. Compute the product of the matrices AB and BA:

(a) $A = \begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix}$;

(b) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$;

(c) $A = \begin{pmatrix} 5 & 8 & -4 \\ 6 & 9 & -5 \\ 4 & 7 & -3 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 2 \\ 4 & 1 \\ 1 & 2 \end{pmatrix}$.

277. Compute the below expressions:

(a) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$; (b) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^3$;

(c) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n$ $(n > 3)$.

278. Find all the matrices $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ permutational (commutative) with the matrix A:

(a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, (b) $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

279. Find the inverse to the given matrix:

(a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, (b) $A = \begin{pmatrix} 2 & 5 & 7 \\ 6 & 3 & 4 \\ 5 & -2 & -3 \end{pmatrix}$;

(c) $A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$.

280. Solve the matrix equation

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} X = \begin{pmatrix} 3 & 5 \\ 5 & 9 \end{pmatrix}, \quad \text{where} \quad X = \begin{pmatrix} x & y \\ z & t \end{pmatrix}.$$

Solution. Let us find the inverse of the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$: $A_{11} = 4$, $A_{12} = -3$, $A_{21} = -2$, $A_{22} = 1$; $\Delta = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = -2$;

$$A^{-1} = \begin{pmatrix} \frac{A_{11}}{\Delta} & \frac{A_{21}}{\Delta} \\ \frac{A_{12}}{\Delta} & \frac{A_{22}}{\Delta} \end{pmatrix} = \begin{pmatrix} -2 & +1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}.$$

Multiplying both sides of the equation by A^{-1}, we get $(A^{-1}A = E)$:

$$X = A^{-1}\begin{pmatrix} 3 & 5 \\ 5 & 9 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 2 & 3 \end{pmatrix}.$$

281. Solve the matrix equation

$$X \cdot \begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -5 & 6 \end{pmatrix}.$$

To find the inverse A^{-1} of the matrix A, we have to solve the system $\mathbf{y} = A\mathbf{x}$ with respect to $\mathbf{x}$. Let $\mathbf{x} = B\mathbf{y}$. Then $A^{-1} = B$, since $AB\mathbf{y} = A\mathbf{x} = \mathbf{y}$, i.e. $AB = E$; $BA\mathbf{x} = B\mathbf{y} = \mathbf{x}$, i.e. $BA = E$, where E is a unit matrix, i.e. $AB = BA = E$.

Let us apply this method to find the inverse of the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. We form the linear system

$$A\mathbf{x} = \mathbf{y} \quad \text{or} \quad \begin{cases} x_1 + 2x_2 = y_1, \\ 3x_1 + 4x_2 = y_2. \end{cases}$$

Solving this system we obtain

$$x_1 = -2y_1 + y_2,$$
$$x_2 = \frac{3}{2} y_1 - \frac{1}{2} y_2.$$

Hence,

$$A^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}.$$

282. Using the indicated method, find A^{-1} for the given matrices:

(a) $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$; (b) $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$.

283. Find out which of the operators in $A\mathbf{x}$ is a linear one, and for a linear operator find its matrix:

(a) $A\mathbf{x} = (x_2 + x_3, 2x_1 + x_3, 3x_1 - x_2 + x_3)$;

(b) $A\mathbf{x} = (x_1, x_2 + 1, x_3 + 1)$.

284. Let in the basis $\mathbf{i}^1, \mathbf{i}^2, \mathbf{i}^3$ there be given linearly independent vectors $\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3$. Find the linear transformation transferring the vectors $\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3$ into $\mathbf{b}^1, \mathbf{b}^2, \mathbf{b}^3$ respectively if

$$\mathbf{a}^1 = (2, 3, 5), \quad \mathbf{a}^2 = (0, 1, 2), \quad \mathbf{a}^3 = (1, 0, 0);$$
$$\mathbf{b}^1 = (1, 1, 1), \quad \mathbf{b}^2 = (1, 1, -1), \quad \mathbf{b}^3 = (2, 1, 2).$$

Hint (see [2], Sec. 16). If we are given the systems of vectors

$$\mathbf{a}^1 = (a_{11}, a_{21}, a_{31}), \quad \mathbf{a}^2 = (a_{12}, a_{22}, a_{32}), \quad \mathbf{a}^3 = (a_{13}, a_{23}, a_{33});$$
$$\mathbf{b}^1 = (b_{11}, b_{21}, b_{31}), \quad \mathbf{b}^2 = (b_{12}, b_{22}, b_{32}), \quad \mathbf{b}^3 = (b_{13}, b_{23}, b_{33}),$$

then the linear operator generated by the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

maps the basis $\mathbf{i}^1$, $\mathbf{i}^2$, $\mathbf{i}^3$ into $\mathbf{a}^1$, $\mathbf{a}^2$, $\mathbf{a}^3$, respectively. Consequently, A^{-1} maps $\mathbf{a}^1$, $\mathbf{a}^2$, $\mathbf{a}^3$ into $\mathbf{i}^1$, $\mathbf{i}^2$, $\mathbf{i}^3$, respectively. Further, the operator B generated by the matrix

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix},$$

maps $\mathbf{i}^1$, $\mathbf{i}^2$, $\mathbf{i}^3$ into $\mathbf{b}^1$, $\mathbf{b}^2$, $\mathbf{b}^3$, respectively. Hence, BA^{-1} maps $\mathbf{a}^1$, $\mathbf{a}^2$, $\mathbf{a}^3$ into $\mathbf{b}^1$, $\mathbf{b}^2$, $\mathbf{b}^3$, respectively.

285. Find the linear transformation mapping the vectors

$$\mathbf{a}^1 = (2, 0, 3), \qquad \mathbf{a}^2 = (4, 1, 5), \qquad \mathbf{a}^3 = (3, 1, 2)$$

into the respective vectors

$$\mathbf{b}^1 = (1, 2, -1), \quad \mathbf{b}^2 = (4, 5, -2), \quad b^3 = (1, -1, 1).$$

286. A linear transformation A has the matrix

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 3 & 0 & -1 & 2 \\ 2 & 5 & 3 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}$$

in the basis $\mathbf{i}^1$, $\mathbf{i}^2$, $\mathbf{i}^3$, $\mathbf{i}^4$. Find the matrix of the same transformation in the basis: (a) $\mathbf{i}^1$, $\mathbf{i}^3$, $\mathbf{i}^2$, $\mathbf{i}^4$; (b) $\mathbf{i}^1$, $\mathbf{i}^1+\mathbf{i}^2$, $\mathbf{i}^1+\mathbf{i}^2+\mathbf{i}^3$, $\mathbf{i}^1+\mathbf{i}^2+\mathbf{i}^3+\mathbf{i}^4$.

287. In the basis $\mathbf{a}^1 = (1, 2)$, $\mathbf{a}^2 = (-1, 1)$ a linear transformation A has the matrix

$$A = \begin{pmatrix} 1 & -2 \\ -1 & -3 \end{pmatrix}.$$

Find the matrix of this transformation in the basis $\mathbf{b}^1 = (1, -2)$, $\mathbf{b}^2 = (3, -1)$.

288. Let transformation A have the matrix $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ in the basis $\mathbf{a}^1$, $\mathbf{a}^2$ (see Problem 287). Find the matrix of this transformation in the basis $\mathbf{b}^1$, $\mathbf{b}^2$.

289. Which of the given pairs of vectors are orthogonal:

(a) $\mathbf{x} = (1, 2, 3)$, $\mathbf{y} = (0, -3, 2)$;

(b) $\mathbf{x} = (1, 2, 1)$, $\mathbf{y} = (0, 1, 2)$;

(c) $\mathbf{x} = (1, 0, 1)$, $\mathbf{y} = (0, 2, 1)$.

290. Show that the system of vectors

$$\mathbf{e}^1 = (1, 2, 3), \quad \mathbf{e}^2 = (0, -3, 2), \quad \mathbf{e}^3 = (13, -2, -3)$$

is an orthogonal basis in R_3. Find the coordinates of the vector $\mathbf{x} = (1, 0, 0)$ in this basis.

291. Supplement an orthonormal system of the vectors

$$\mathbf{x} = (1/\sqrt{2}, 0, 1/\sqrt{2}, 0), \quad \mathbf{y} = (0, -1/\sqrt{2}, 0, 1/\sqrt{2})$$

with the vectors $\mathbf{z}$ and $\mathbf{t}$ to complete an orthonormal basis in R_4.

292. Find out the orientation of the orthogonal basis with respect to the main basis $\mathbf{i}^1 = (1, 0, 0)$, $\mathbf{i}^2 = (0, 1, 0)$, $\mathbf{i}^3 = (0, 0, 1)$:

(a) $\mathbf{a}^1 = \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$, $\mathbf{a}^2 = \left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$,

$\mathbf{a}^3 = \left(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$;

(b) $\mathbf{a}^1 = \left(\frac{1}{2}, \frac{1}{2}, -\frac{\sqrt{2}}{2}\right)$, $\mathbf{a}^2 = \left(\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}\right)$,

$\mathbf{a}^3 = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0\right)$.

293. Let a new orthogonal basis $\mathbf{b}^1$, $\mathbf{b}^2$ be given by the orthogonal matrix

$$A = \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix}.$$

Write the formulas relating the coordinates (x_1, x_2) of the vector $\mathbf{a}$ in the old basis with its coordinates (x_1', x_2') in the new basis.

294. Let the basis $\mathbf{a}^1$, $\mathbf{a}^2$ be given by the matrix $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$. Write the transformation formulas for passing from the coordinates of the vector $\mathbf{a}$ in the old basis to the coordinates in the new basis, and vice versa.

Sec. 3.11. Linear Subspaces (See [2], Sec. 20)

295. Is the following set of vectors a linear subspace:

(a) having odd integral coordinates;
(b) having even integral coordinates;
(c) lying on a straight line passing through the origin;
(d) lying on the x- or y-axis;
(e) whose terminal points lie in the first quadrant of the coordinate system (the initial points of the vectors are supposed to be coincident with the origin);
(f) whose terminal points lie on a given straight line;
(g) whose terminal and initial points lie on a given line;
(h) which are all possible linear combinations of the vectors $x^1, x^2, \ldots, x^k$ in R_n $(k \leqslant n)$?

296. Enumerate all linear subspaces of R_2.

297. Let L be a subspace of R_2 (that is, the set of vectors lying on the straight line $x_2 = kx_1$). Find the subspace L' which is orthogonal to the given one.

298. Find the dimension and basis of linear subspaces which are linear combinations of the indicated vectors (or, as we say, *spanned on a given system of vectors*):

(a) $\mathbf{a}^1 = (1, 0, 0, -1)$, $\mathbf{a}^2 = (2, 1, 1, 0)$, $\mathbf{a}^3 = (1, 1, 1, 1)$, $\mathbf{a}^4 = (1, 2, 3, 4)$, $\mathbf{a}^5 = (0, 1, 2, 3)$;

(b) $\mathbf{a}^1 = (1, 0, 1)$, $\mathbf{a}^2 = (1, 1, 1,)$ $\mathbf{a}^3 = (2, 1, 2)$, $\mathbf{a}^4 = (3, 2, 3)$.

299. Let L be a subspace in R_4 spanned on the vectors

$$\mathbf{a}^1 = (1, 0, 0, -1), \quad \mathbf{a}^2 = (2, 1, 1, 0).$$

Find the subspace L' orthogonal to L. Let the vector $\mathbf{a}$ be orthogonal to L'. Prove that this vector is a linear combination of the vectors $\mathbf{a}^1$, $\mathbf{a}^2$ ($\mathbf{a} = \alpha\mathbf{a}^1 + \beta\mathbf{a}^2$).

300. Let $\mathbf{e}^1$, $\mathbf{e}^2$ be an orthonormal basis in a plane and linear operator A has the matrix $\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$ in the basis $\mathbf{f}^1 = \mathbf{e}^1$, $\mathbf{f}^2 = \mathbf{e}^1 + \mathbf{e}^2$. Find the matrix of the adjoint operator A^* in the same basis $\mathbf{f}^1$, $\mathbf{f}^2$.

Solution. In the basis $\mathbf{e}^1$, $\mathbf{e}^2$ the matrix of operator A^* is a transpose of the matrix of operator A.

Let us first find the matrix of operator A in the basis $\mathbf{e}^1$, $\mathbf{e}^2$. We have

$$A(\mathbf{e}^1) = A(\mathbf{f}^1) = \mathbf{f}^1 + \mathbf{f}^2 = 2\mathbf{e}^1 + \mathbf{e}^2;$$

$$A(\mathbf{e}^2) = A(\mathbf{f}^2 - \mathbf{f}^1) = A(\mathbf{f}^2) - A(\mathbf{f}^1) = \mathbf{f}^1 - 2\mathbf{f}^2 = -\mathbf{e}^1 - 2\mathbf{e}^2.$$

Thus, the matrix of operator A in the basis $\mathbf{e}^1$, $\mathbf{e}^2$ has the form

$$A = \begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix}, \quad A^* = \begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix}.$$

Now we find the value of operator A^* on the vectors $\mathbf{f}^1$, $\mathbf{f}^2$:

$$A^*\mathbf{f}^1 = A^*\mathbf{e}^1 = 2\mathbf{e}^1 - \mathbf{e}^2 = 3\mathbf{f}^1 - \mathbf{f}^2,$$

$$A^*\mathbf{f}^2 = A^*(\mathbf{e}^1 + \mathbf{e}^2) = A^*\mathbf{e}^1 + A^*\mathbf{e}^2 = 3\mathbf{f}^1 - \mathbf{f}^2 + \mathbf{e}^1 - 2\mathbf{e}^2 = 6\mathbf{f}^1 - 3\mathbf{f}^2.$$

Hence, the matrix of A^* in the basis $\mathbf{f}^1$, $\mathbf{f}^2$ has the form

$$\begin{pmatrix} 3 & 6 \\ -1 & -3 \end{pmatrix}.$$

301. Let in Problem 300 the matrix of operator A in the basis $\mathbf{f}^1$, $\mathbf{f}^2$ be equal to $\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$. Find the matrix of A^* in the same basis.

302. A linear operator A in the basis $\mathbf{f}^1 = (1, 2, 1)$, $\mathbf{f}^2 = (1, 1, 2)$, $\mathbf{f}^3 = (1, 1, 0)$ is given by the matrix

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & 5 & -1 \\ 2 & 7 & -3 \end{pmatrix}.$$

Find the matrix of A^* in the same basis regarding that the coordinates of the vectors are given in some orthonormal basis (for instance, $\mathbf{f}^1 = \mathbf{e}^1 + 2\mathbf{e}^2 + \mathbf{e}^3$).

Sec. 3.12. Self-adjoint Operators. Quadratic Forms (See [2], Secs. 22, 23)

303. Find the greatest eigenvalue of the self-adjoint operator defined by the matrix:

(a) $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$; (b) $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

304. Using Sylvester's theorem, find out whether the given quadratic form is strictly positive:

(a) $x_1^2 + x_2^2 + 3x_3^2 + 4x_1x_2 + 2x_1x_3 + 2x_2x_3$;

(b) $2x_1x_2 + 2x_1x_3 + 2x_1x_4 + 2x_2x_3 + 2x_2x_4 + 2x_3x_4$;

(c) $2x_1^2 + x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3$.

305. By finding the eigenvalues, find out the type of the quadratic form:

(a) $f = x^2 + 4xy - y^2$;

(b) $f = x^2 + 26y^2 + 10xy$;

(c) $f = x^2 + 3y^2 + 2\sqrt{3}xy$.

306. Reduce the indicated quadratic form to the canonical form:

(a) $3x_2^2 + 3x_3^2 + 4x_1x_2 + 4x_1x_3 - 2x_2x_3$;

(b) $7x_1^2 + 7x_2^2 + 7x_3^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3$.

307. Find the orthogonal transformation which reduces the following forms to the canonical form. Write this canonical form:

(a) $6x_1^2 + 5x_2^2 + 7x_3^2 - 4x_1x_2 + 4x_1x_3$;

(b) $17x_1^2 + 14x_2^2 + 14x_3^2 - 4x_1x_2 - 4x_1x_3 - 8x_2x_3$.

Sec. 3.13. Second-order Curves (See [2], Sec. 24)

The general equation of a second-order curve is written as

$$Ax^2+2Bxy+Cy^2+Dx+Ey+F=0, \tag{1}$$

where A, B, C are not all zero.

We assume that $B \geqslant 0$. Equation (1) can also be reduced to this by setting $x=x'$, $y=-y'$. An appropriate orthogonal transformation reduces equation (1) to the form

$$\lambda_1\xi^2+\lambda_2\eta^2+2d\xi+2e\eta+g=0, \tag{2}$$

where λ_1, λ_2 are the eigenvalues of the linear self-adjoint operator generated by the quadratic form

$$Ax^2+2Bxy+Cy^2 \tag{3}$$

and d, e, g are certain numbers. For $B=0$ equation (1) has form (2). Therefore, further on we assume that $B>0$.

The coordinates ξ, η are considered in a new rectangular system (new orthonormal basis) whose unit vectors are the eigenvectors of the indicated self-adjoint operator. In this case, if the first eigenvector $\mathbf{x}^1$ (unit vector), corresponding to the eigenvalue λ_1, has the coordinates (x_0, y_0), then for the second eigenvector (for $B>0$) we take the vector $(-y_0, x_0)$ or $(y_0, -x_0)$. Note that the vectors (x_0, y_0), $(-y_0, x_0)$ are oriented in the same way as the initial basis $\mathbf{i}=(1, 0)$, $\mathbf{j}=(0, 1)$, whereas the vectors (x_0, y_0), $(y_0, -x_0)$ are oriented in the opposite way $\left(\Delta=\begin{vmatrix} x_0 & y_0 \\ y_0 & -x_0 \end{vmatrix}=-1<0\right)$.

Thus, if $B>0$, then the coordinates are transformed to the form

$$\begin{cases} x = x_0\xi \mp y_0\eta, \\ y = y_0\xi \pm x_0\eta. \end{cases}$$

In the two-dimensional case the eigenvalues and eigenvectors can be computed by the formulas obtained in [2], Sec. 24:

$$\lambda_1=\frac{A+C}{2}+\frac{1}{2}\sqrt{4B^2+(A-C)^2},$$

$$\lambda_2=\frac{A+C}{2}-\frac{1}{2}\sqrt{4B^2+(A-C)^2};$$

$$x_0=\sqrt{\frac{1}{2}+\frac{A-C}{2\sqrt{4B^2+(A-C)^2}}},$$

$$y_0=\sqrt{\frac{1}{2}-\frac{A-C}{2\sqrt{4B^2+(A-C)^2}}}.$$

Of course, each time the eigenvalues can be found as the roots of the characteristic equation

$$\begin{vmatrix} A-\lambda & B \\ B & C-\lambda \end{vmatrix} = 0,$$

and the coordinates of the eigenvector $\mathbf{x}^1$ as the solution of the system

$$\begin{cases} (A-\lambda_1)x_0+ & By_0 = 0, \\ Bx_0 & +(C-\lambda_1)y_0 = 0. \end{cases}$$

If both eigenvalues λ_1 and λ_2 are not equal to zero, then equation (2) can be written in the following way:

$$\lambda_1(\xi-\alpha)^2+\lambda_2(\eta-\beta)^2 = \gamma, \quad \alpha = -\frac{d}{\lambda_1}, \quad \beta = -\frac{e}{\lambda_2}, \qquad (4)$$

where γ is a constant. Whence, setting $u = \xi-\alpha$, $v = \eta-\beta$, we get

$$\lambda_1 u^2+\lambda_2 v^2 = \gamma. \qquad (5)$$

If $AC-B^2 > 0$, then $\lambda_1\lambda_2 > 0$, and (5) directly yields the canonical equation of an ellipse (real or imaginary) or a point.

If $AC-B^2 < 0$, then $\lambda_1\lambda_2 < 0$ and (5) readily yields the canonical equation of a hyperbola or a pair of intersecting straight lines.

And if $AC-B^2 = 0$, then $\lambda_1\lambda_2 = 0$. But if one of the eigenvalues, say λ_1, is different from zero, then equation (2) is written in the form

$$\lambda_1(\xi-\alpha)^2+\delta\eta = \omega. \qquad (6)$$

If $\delta = 0$, then equation (6) defines a pair of straight lines (real or imaginary). But if $\delta \neq 0$, then, setting $u = \xi-\alpha$, $v = \eta-\frac{\omega}{\delta}$ and, possibly, $v = -v'$, we obtain the canonical equation of parabola.

308. Find out the type of the following curves and reduce their equations to the canonical form:

(a) $3x^2+3y^2-6x-12y+3 = 0$;

(b) $3x^2+2y^2-6x-12y+15 = 0$;

(c) $x^2-2y^2+4y-4 = 0$;

(d) $3x^2-6x+3y^2-12y+15 = 0$;

(e) $x^2-2y^2+4y-2 = 0$;

(f) $4x-3y^2+12y-12 = 0$;

(g) $3x^2+2y^2-6x-12y+22 = 0$.

309. Determine the type of the following curves and reduce their equations to the canonical form. Write the transformations of the coordinate system. Represent the coordinate systems and the curves.

(a) $3x^2+10xy+3y^2-2x-14y-13=0$;

(b) $25x^2-14xy+25y^2+64x-64y-224=0$;

(c) $9x^2-24xy+16y^2-20x+110y-50=0$.

310. Given the equation of a curve: $4x^2-4xy+y^2+6x+1=0$. Determine for what values of k the straight line $y=kx$

(a) has one common point with the curve;

(b) intersects the curve at two points;

(c) has no common points with the curve.

311. For what values of k does the straight line $y=kx$ touch the curve $(x+y)^2+2=\sqrt{2}(y-x)$.

312. Write the equation of the second-order curve passing through the following points: (0, 0), (1, 0), (1, 1), (−2, 1), (0, 3).

Sec. 3.14. Second-order Surfaces (See [2], Sec. 25)

The general equation of a second-order surface has the form

$$a_{11}x_1^2+a_{22}x_2^2+a_{33}x_3^2+2a_{12}x_1x_2+2a_{13}x_1x_3+2a_{23}x_2x_3 +2A_1x_1+2A_2x_2+2A_3x_3+B=0. \quad (1)$$

If the mixed products of the variables are absent from equation (1) (i.e. $a_{12}=a_{13}=a_{23}=0$), then the equation of a surface is reduced to the canonical form by separating perfect squares with respect to the variables x_1, x_2, x_3 of the form

$$a_{11}(x_1-\alpha_1)^2+a_{22}(x_2-\alpha_2)^2+a_{33}(x_3-\alpha_3)^2$$

and translating the origin to the point $(\alpha_1, \alpha_2, \alpha_3)$.

But if the mixed products are present, then we first reduce the symmetric quadratic form $\sum_{k=1}^{3}\sum_{l=1}^{3}a_{kl}x_kx_l$ to the canonical form.

The eigenvalues of the operator generated by this quadratic form are found as the roots of the characteristic equation

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix}=0,$$

and the coordinates of the eigenvectors $\mathbf{x}^k$ ($k = 1, 2, 3$) are found from the systems

$$\begin{cases} (a_{11}-\lambda_k)x_1+ & a_{12}x_2 & + & a_{13}x_3 & = 0, \\ a_{21}x_1 & +(a_{22}-\lambda_k)x_2+ & & a_{23}x_3 & = 0, \\ a_{31}x_1 & + & a_{32}x_2 & +(a_{33}-\lambda_k)x_3 & = 0, \end{cases}$$

which, as it is shown in [2], Sec. 25, always have solutions (three pairwise orthogonal vectors $\mathbf{x}^1$, $\mathbf{x}^2$, $\mathbf{x}^3$).

313. Reduce the indicated equation to the canonical form and name the surface defined by this equation:

(a) $x^2+y^2+z^2+2x+4y-4=0$;

(b) $x^2+2y^2+z^2+2x+4y-1=0$;

(c) $x^2+2y^2-z^2+2x+4y-1=0$;

(d) $x^2+2y^2+2x+4y-2z+3=0$;

(e) $x^2-4y^2-z^2+8y-2z-9=0$;

(f) $x^2+2y^2-2x-4y-1=0$.

314. Reduce the given equations of surfaces to the canonical form and indicate the appropriate transformations of the coordinates

(a) $11x_1^2+5x_2^2+2x_3^2+16x_1x_2+4x_1x_3-20x_2x_3+2x_1+2x_2+2x_3+1=0$;

(b) $3x_2^2+3x_3^2+4x_1x_2+4x_1x_3-2x_2x_3+4x_1+1=0$.

315. Find the curve along which the plane $x = 2$ cuts the ellipsoid $\frac{x^2}{16}+\frac{y^2}{12}+\frac{z^2}{4}=1$.

316. Write the canonical equation of the hyperboloid of one sheet passing through the points (1, 0, 0), (0, 4, 0), (1, 1, 1).

317. Write the equations of the projections (on the coordinate planes) of the section of the elliptic paraboloid $y^2+z^2=x$ by the plane $x+2y-z=0$.

318. What line is defined by the equations

$$\begin{cases} \frac{x^2}{4}-\frac{y^2}{3}=2z, \\ x-2y+2=0? \end{cases}$$

319. Write the equation of the tangent plane to the hyperboloid of one sheet $\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ at the point $(0, 0, c)$.

320. Write the equation of the surface generated by revolving the ellipse

$$\begin{cases} \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1, \\ z=0 \end{cases}$$

about the x-axis.

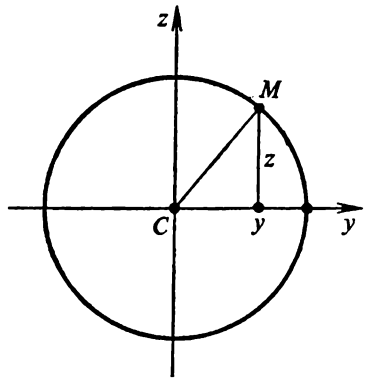

Fig. 18.

Solution. Let us take an arbitrary point $P=(\bar{x}, \bar{y}, 0)$ on the given ellipse. During rotation of the ellipse about the x-axis the point P describes a circle of radius $\bar{y}$. Let $M=(x, y, z)$ be an arbitrary point on this circle (and, consequently, on the desired surface). It is obvious that $CP = |\bar{y}| = CM = \sqrt{y^2+z^2}$, $x=\bar{x}$ (see Fig. 18). Since the point P lies on the ellipse, we have

$$\frac{\bar{x}^2}{a^2}+\frac{\bar{y}^2}{b^2}=1.$$

Replacing $\bar{y}$ and $\bar{x}$ by their values, we obtain

$$\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1.$$

This is just the sought-for equation of the generated surface.

Remark 1. Since the curve $f(x, y)=0$ is revolved about the x-axis, the y-coordinate is replaced by $\sqrt{y^2+z^2}$ in the equation of this curve. As a result, we get the equation of a surface of revolution about the x-axis:

$$f(x, \sqrt{y^2+z^2})=0.$$

It is also obvious that

$$f(\sqrt{x^2+z^2}, y)=0$$

is the equation of a surface of revolution about the y-axis.

321. Write the equation of the surface of revolution generated by rotating the curve

(a) $\begin{cases} x^2=2z, \\ y=0; \end{cases}$ (b) $\begin{cases} \dfrac{x^2}{a^2}+\dfrac{z^2}{c^2}=1, \\ y=0 \end{cases}$

about the z-axis.

322. Find the points of intersection of the folloving surface and straight line:

(a) $\frac{x^2}{81}+\frac{y^2}{36}+\frac{z^2}{9}=1, \quad \frac{x-3}{3}=\frac{y-4}{-6}=\frac{z+2}{4};$

(b) $\frac{x^2}{5}+\frac{y^2}{3}=z, \quad \frac{x+1}{2}=\frac{y-2}{-1}=\frac{z+3}{-2}.$

323. Write the equation of the cone with vertex at the origin whose elements touch the sphere

$$(x+2)^2+(y-1)^2+(z-3)^2=9.$$

Solution. The given sphere is of radius 3, its centre C having the coordinates $C=(-2, 1, 3)$. Hence, the xy-plane touches the sphere at the point $P=(-2, 1, 0)$ (Fig. 19). The ray OP is the generatrix of the cone. The directrix of the cone lies on the sphere and in the plane passing through the point P perpendicular to the straight line OC. The equation of the straight line OC is

$$\frac{X}{-2}=\frac{Y}{1}=\frac{Z}{3}.$$

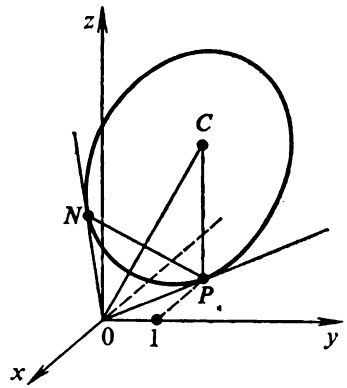

Fig. 19.

The equation of the plane passing through the point P perpendicular to OC is

$$-2(x+2)+(y-1)+3z=0.$$

Therefore, the directrix of the cone can be defined by the system

$$\begin{cases}(x+2)^2+(y-1)^2+(z-3)^2=9,\\ -2x+y+3z-5=0.\end{cases}$$

Let now (x, y, z) be an arbitrary point on the directrix and let (X, Y, Z) be an arbitrary point of the generatrix (and, consequently, a point belonging to the cone). The equation of the generatrix can be written as the equation of the straight line passing through the points $(0, 0, 0)$ and (x, y, z):

$$\frac{X}{x}=\frac{Y}{y}=\frac{Z}{z}.$$

Eliminating x, y, z from the system and from the last three equations, we obtain the equation of the cone. We then fix $z = c$ and express x and y in terms of X, Y, Z:

$$x = \frac{cX}{Z}, \quad y = \frac{cY}{Z}.$$

Substituting these values into the system and eliminating the parameter c, after elementary transformations we obtain the required equation of the cone:

$$X^2+4Y^2-4Z^2+4XY+12XZ-6YZ = 0.$$

324. Write the equation of the cone with vertex $S = (5, 0, 0)$ whose elements touch the sphere $x^2+y^2+z^2 = 9$.

Chapter 4

FUNCTIONS OF SEVERAL VARIABLES
(See [1], Chapter 8)

Sec. 4.1. Basic Concepts

325. Find and represent the domains of existence of the following functions:

(a) $u = \sqrt{1-x^2-4y^2}$; (b) $u = \sqrt{1-\frac{x^2}{9}+\frac{y^2}{4}}$;

(c) $u = \sqrt{y^2-4x}$; (d) $u = \frac{1}{x^2+y^2}$;

(e) $u = \ln(x+y)$; (f) $u = \arcsin\frac{y}{x}$.

326. Find and represent the domains of existence of the following functions of three variables:

(a) $u = \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}$; (b) $u = \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}}$;

(c) $u = \ln(-x^2-y^2+2z)$; (d) $u = \sqrt{1+\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}}$;

(e) $u = \arcsin x + \arcsin y + \arcsin z$.

327. Evaluate the function

$$f(x, y) = x^2+\frac{y}{x}$$

at the points (1, 0), (1, 1), (2, 1).

328. Find $f(x, y)$ if $f(x+2y, x-2y) = xy$.

329. The *contour* (or level) *line* of the function $u = f(x, y)$ is defined as the set of points belonging to the domain of its definition at which it attains the prescribed constant value: $f(x, y) = c$. The latter is thus the equation of the contour line.

Geometrically, it means that we cut the surface defined by this function (i.e. its graph) by the plane $u = c$ and projected the line thus obtained on the xy-plane. This projection of the section is just the contour line.

Find the contour lines of the given functions:

(a) $u = 1-\frac{x^2}{a^2}-\frac{y^2}{b^2}$; (b) $u = \frac{y}{x^2}$.

330. Find the distance ϱ between the points (1, 0, 1) and (2, 1, 0) in space R_3.

331. Find the limit of the sequence of points

$$M^k = \left(\frac{1}{1+k}, \frac{k^2}{k^2+1}\right) \qquad (k = 1, 2, \ldots).$$

332. Let the set $E = \{|x| < 1, |y| \leqslant 1\}$. Which points of this set are interior?

333. Are the following sets connected ones:

(a) $E = \{|x|+|y| \leqslant 1\}$; (b) $E = \left\{\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} \geqslant 1\right\}$;

(c) $E = \{x^2+y^2 \neq 1\}$?

Sec. 4.2. Limit of a Function. Continuity (See [1], Secs. 8.3, 8.4)

A function f has a limit at a point $\mathbf{x}^0$ equal to A if it is defined in some neighbourhood of the point $\mathbf{x}^0$ except, possibly, at the point $\mathbf{x}^0$ itself, and if

$$\lim_{\substack{\mathbf{x}^k \to \mathbf{x}^0 \\ \mathbf{x}^k \neq \mathbf{x}^0}} f(\mathbf{x}^k) = A,$$

whatever the sequence of points $\mathbf{x}^k$ from the mentioned neighbourhood (different from $\mathbf{x}^0$) tending to $\mathbf{x}^0$.

But there are cases when a function f is defined not in the entire neighbourhood, but only in one of its subsets E. In this case there arises the notion of the limit of a function at the point $\mathbf{x}^0$ with respect to the set E.

The number A is said to be the limit of the function f at the point $\mathbf{x}^0 \in \bar{E}$ ($\bar{E}$ is the closure of E, see [1], Sec. 8.11) with respect to the set E if

$$\lim_{\substack{\mathbf{x}^k \to \mathbf{x}^0 \\ \mathbf{x}^k \in E,\ \mathbf{x}^k \neq \mathbf{x}^0}} f(\mathbf{x}^k) = A,$$

whatever the sequence of points $\mathbf{x}^k \in E$, converging to $\mathbf{x}^0$. This definition is equivalent to the following: the number A is called the limit of a function f at point $\mathbf{x}^0 \in \bar{E}$ with respect to the set E if $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon, \mathbf{x}^0) > 0$ such that

$$|f(\mathbf{x}) - A| < \varepsilon, \quad \forall x \in E, \quad 0 < |\mathbf{x} - \mathbf{x}^0| < \delta.$$

Example 1. The function

$$f(x) = f(x_1, x_2) = \sqrt{x_1^2 + x_2^2} \sin \frac{1}{x_1^2 + x_2^2}$$

is defined in the entire plane R_2 except the origin. It is obvious that in any neighbourhood of the origin the function f satisfies the inequality

$$|f(\mathbf{x})| = \left|\sqrt{x_1^2+x_2^2}\sin\frac{1}{x_1^2+x_2^2}\right| \leqslant \sqrt{x_1^2+x_2^2}$$
$$= |\mathbf{x}| = |\mathbf{x}-\mathbf{0}| < \varepsilon \quad (\mathbf{x} \neq 0)$$

provided that $|\mathbf{x}| < \delta = \varepsilon$. Hence, at the point $\mathbf{0} = (0, 0)$ an ordinary limit exists and is equal to zero:

$$\lim_{\substack{x \to 0 \\ x \neq 0}} f(\mathbf{x}) = 0.$$

Example 2. The function

$$f(x_1, x_2) = (x_1^2+x_2^2)\sin\frac{1}{x_1 x_2}$$

is defined on the set E representing the plane R_2 without the coordinate axes. The function f has no ordinary limit at the point $\mathbf{0} = (0, 0)$, but the limit of f at this point with respect to E exists and is equal to zero

$$\lim_{\substack{x_1 \to 0,\ x_2 \to 0 \\ (x_1, x_2) \in E}} f(x_1, x_2) = 0.$$

Problems. Consider the existence of the limit of the given function at the indicated point:

(a) $f(x, y) = \dfrac{\sin(x^2+y^2)}{x^2+y^2}$ at the point $(0, 0)$;

(b) $f(x, y) = \dfrac{1-\cos(x^2+y^2)}{(x^2+y^2)x^2y^2}$ at the point $(0, 0)$;

(c) $f(x, y, z) = \exp(-1/(x^2+y^2+z^2))/(x^4+y^4+z^4)$ at the point $(0, 0, 0)$.

334. Find the limit of the given function:

(a) $\lim\limits_{\substack{x \to 0 \\ y \to 2}} \dfrac{\sin xy}{x}$ $(x \neq 0)$; (b) $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \dfrac{x^2-y^2}{x^2+y^2}$.

335. For what c will the function

$$f(x, y) = \begin{cases} \sqrt{1-x^2-4y^2}, & x^2+4y^2 \leqslant 1, \\ c, & x^2+4y^2 > 1 \end{cases}$$

be continuous in the entire xy-plane?

336. Find out: (a) whether the function

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & x^2+y^2 \neq 0, \\ 0, & x = y = 0 \end{cases}$$

is continuous at the point (0, 0);

(b) whether it is continuous on the ray in the direction of any vector $\boldsymbol{\omega} \neq \mathbf{0}$, emanating from the origin.

337. Prove that the set of points (x, y) satisfying the inequality $1-x^2-y^2 > c$ is an open set.

Sec. 4.3. Partial Derivatives. Differentials (See [1], Secs. 8.5 and 8.6)

In Problems 338 to 341 find the partial derivatives of the given functions and their total differentials.

338. $u = x^3+y^2-2xy$. **339.** $u = x^2y^3$.

340. $u = \ln(x+\sqrt{x^2+y^2})$.

341. (a) $u = \arctan(y/x)$; (b) $u = xy+\dfrac{x}{y}$;

(c) $u = x^y$; (d) $u = \sinh(x+y)$; (e) $u = \cosh(x^2y+\sinh y)$.

342. Compute the determinant

$$\Delta = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \varphi} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \varphi} \end{vmatrix}$$

if: (a) $x = r\cos\varphi$, $y = r\sin\varphi$; (b) $x = r^2+\varphi$, $y = r+\varphi^2$.

343. Find the partial derivatives of the indicated composite functions with respect to the variables t and τ:

(a) $u = \sqrt{x+y}$, where $x = e^{t+\tau}$, $y = \ln t$;

(b) $u = xy$, where $x = \cos(t+\tau)$, $y = \sin(t-\tau)$.

344. Find and construct the gradient of the given functions at the point $P = (1, 1)$:

(a) $u = x^2y$; (b) $u = 2x^2-3y^2$.

345. Find at the point $P = (1, 1)$ the derivatives of the functions u in the direction of the vector $\mathbf{n} = (\sqrt{3}/2, 1/2)$:

(a) $u = \ln\sqrt{x^2+y^2}$; (b) $u = 2x^2-3y^2$.

346. Find the derivative of the function $u = 2x^2 - 3y^2$ at the point $P = (1, 1)$ in the direction of the gradient.

347. Find the angles made by the gradient of the given function at the point $P = (1, 1)$ with the coordinate axes:

(a) $u = x^{\sqrt{3}} + y$; (b) $u = x + y^{\sqrt{3}}$.

Sec. 4.4. Partial Derivatives and Differentials of Higher Orders (See [1], Secs. 8.5 and 8.9)

348. Find the partial derivatives and second-order differentials of the following functions:

(a) $u = \ln(x^2 + y)$; (b) $u = \sqrt{2xy + y^2}$.

349. Show that the indicated functions

(a) $u = \arctan(y/x)$; (b) $u = -\ln\sqrt{(x-a)^2 + (y-b)^2}$ satisfy Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

350. Show that the function

$$u = \varphi(x - at) + \psi(x + at),$$

where φ, ψ have derivatives up to the second order, satisfies the equation $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$.

351. Find the derivatives and differentials of the second order of the composite functions (x, y are independent variables):

(a) $u = f(\xi, \eta)$, $\xi = ax$, $\eta = by$;

(b) $u = f(\xi, \eta)$, $\xi = x + y$, $\eta = x - y$.

It is supposed that $f(\xi, \eta)$ has derivatives up to the second order (inclusive) with respect to all variables.

Sec. 4.5. Tangent Plane and a Normal to a Surface (See [1], Sec. 8.7)

352. Write the equation of the tangent plane and the equation of the normal to the given surface at the indicated point:

(a) to the paraboloid of revolution $u = x^2 + y^2$ at the point (1, 2, 5);

(b) to the surface $u = \frac{x^2}{2} - y^2$ at the point $(2, -1, 1)$.

Sec. 4.6. Taylor's Formula (See [1], Sec. 8.10)

353. Find the increment in the function:

(a) $u = x^2-y^2+xy$ when passing from the values $x = 1$, $y = 2$ to the values $x_1 = 1+h$, $y_1 = 2+k$;

(b) $u = x^2y$ when passing from the values $x = 1$, $y = 1$ to the values $x_1 = 1+h$, $y_1 = 1+k$.

354. Using Taylor's formula, expand the function $f(x, y) = e^x \sin y$ in the neighbourhood of the point (0, 0) up to the terms of the third order inclusive.

355. Using Taylor's formula, expand the function $f(x, y) = \exp(x+y)$ up to the terms of the third order (inclusive) in the neighbourhood of the point $(1, -1)$.

356. Find the value of the parameter θ in Lagrange's formula for functions of two variables (see [1], Sec. 8.10):

$$f(\mathbf{x})-f(\mathbf{x}^0) = \left(\frac{\partial f}{\partial x_1}\right)_{\mathbf{x}^0+\theta(\mathbf{x}-\mathbf{x}^0)}(x_1-x_1^0) + \left(\frac{\partial f}{\partial x_2}\right)_{\mathbf{x}^0+\theta(\mathbf{x}-\mathbf{x}^0)}(x_2-x_2^0) \quad (0 < \theta < 1);$$

(a) $f(\mathbf{x}) = x_1^2+x_2^2$ with respect to the points $\mathbf{x}^0 = (0, 0)$, $\mathbf{x} = (1, 1)$;

(b) $f(\mathbf{x}) = x_1^2+x_2^3$ with respect to the same points.

Sec. 4.7. Extrema (See [1], Sec. 8.13)

In Problems 357 to 361 investigate the given functions for extremum.

357. $z = (x-2)^2+2y^2$. **358.** $z = (x-2)^2-2y^2$.

359. $z = x^4+4xy-2y^2$.

360. $z = x^4+y^4-2x^2+4xy-2y^2$.

361. $u = x^2+y^2+z^2-xy+x-2z$.

362. Find out whether the given function has the greatest value within the indicated intervals. Find it if any:

(a) $z = \begin{cases} 1, & 0 \leqslant x \leqslant 1, \quad 0 \leqslant y \leqslant 1, \\ 2(2-y), & 0 \leqslant x \leqslant 1, \quad 1 < y \leqslant 2; \end{cases}$

(b) $z = \dfrac{x^2+y}{x^2+y^2+1}$ $(|x| \leqslant 1, \ |y| \leqslant 1)$.

Sec. 4.8. Implicit Functions. Conditional Extremum (See [1], Secs. 8.15-8.17)

363. Find the derivatives y'_x, y''_{x^2} of the implicit function $y(x)$ given by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

364. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, if

$$x\cos y+y\cos z+z\cos x=1.$$

365. $F(x, y, z)=0$. Prove that

$$\frac{\partial x}{\partial y}\cdot\frac{\partial y}{\partial x}=1, \quad \frac{\partial y}{\partial z}\cdot\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial y}=-1.$$

366. Functions u, v of the variables x, y are represented implicitly by the system of equations

$$\begin{cases} x-\varphi(u, v)=0, \\ y-\psi(u, v)=0. \end{cases}$$

Find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$.

367. Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ if $x=u\cos v$, $y=u\sin v$, $z=cv$.

368. Write the equation of the tangent plane to the indicated surface:

(a) $x^2+2y^2+3z^2=21$ at the point $(\sqrt{3}, 0, 6)$;

(b) to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ at its point (x_0, y_0, z_0).

369. To the surface $x^2+2y^2+3z^2=21$ draw the tangent planes parallel to the plane $x+4y+6z=0$.

370. Write the equations of the tangent plane and the normal to the surface $3xyz-z^3=a^3$ at its point $(0, a, -a)$.

371. Find the rectangle of the greatest area having the given perimeter l.

372. Find the axes of the ellipse $5x^2+8xy+5y^2=9$.

373. Of all the triangles of the given perimeter $2l$ find the one having the greatest area.

Chapter 5

SERIES
(See [1], Chapter 9)

Sec. 5.1. Number Series
(See [1], Sec. 9.1)

374. Using the definition, consider the convergence of the given series and find their sums:

(a) $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n(n+1)}+\ldots$;

(b) $\frac{1}{2\cdot 5}+\frac{1}{3\cdot 6}+\ldots+\frac{1}{(n+1)(n+4)}+\ldots$;

(c) $\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}$.

375. Prove that the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent, applying the integral test and Cauchy's criterion.

376. Using the integral test for convergence of a series, find out for what $\alpha > 0$ the series $\sum_{n=1}^{\infty} n^{-\alpha}$ is convergent. Prove that

$$S_N^1 = \sum_{k=1}^{N}\frac{1}{k} = O(\ln(N+1)), \quad N \geqslant 2; \tag{1}$$

$$S_N^{\alpha} = \sum_{k=1}^{N}\frac{1}{k^{\alpha}} = O[(N+1)^{1-\alpha}], \quad 0 < \alpha < 1, \quad N \geqslant 1; \tag{2}$$

$$R_N^{\alpha} = \sum_{k=N}^{\infty}\frac{1}{k^{\alpha}} = O(N^{1-\alpha}), \quad \alpha > 1, \quad N \geqslant 1. \tag{3}$$

Solution. Since the function $f(x) = x^{-\alpha}$ ($\alpha > 0$) is monotonically decreasing to zero on $(0, \infty)$ as $x \to \infty$, the series $\sum k^{-\alpha}$ and the improper integral $\int_1^{\infty} x^{-\alpha}\,dx$ are simultaneously convergent or divergent.

As is known, this improper integral converges for $\alpha > 0$ and diverges for $0 < \alpha \leqslant 1$, hence, the series $\sum k^{-\alpha}$ also converges for $\alpha > 1$ and diverges for $0 < \alpha \leqslant 1$.

Let us now estimate the growth of S_N^{α} ($0 < \alpha \leqslant 1$). Let $\alpha = 1$;

then

$$\ln(N+1) = \int_1^{N+1} \frac{dx}{x} = \sum_{k=1}^{N} \int_k^{k+1} \frac{dx}{x} \leqslant \sum_{k=1}^{N} \frac{1}{k} = S_N^1 \quad (N \geqslant 1);$$

$$\ln(N+1) = \int_1^{N+1} \frac{dx}{x} = \sum_{k=1}^{N} \int_k^{k+1} \frac{dx}{x} \geqslant \sum_{k=1}^{N} \frac{1}{k+1}$$

$$= \sum_{k=2}^{N+1} \frac{1}{k} = S_N^1 - 1 + \frac{1}{N+1}.$$

Hence,

$$\ln(N+1) \leqslant S_N^1 \leqslant 1 + \ln(N+1) \leqslant 2\ln(N+1), \quad N \geqslant 2.$$

The inequality $|S_N^1| \leqslant 2\ln(N+1)$ just proves property (1).

Using the equality $(0 < \alpha < 1)$

$$\frac{(N+1)^{1-\alpha}}{1-\alpha} - \frac{1}{1-\alpha} = \int_1^{N+1} \frac{dx}{x^\alpha} = \sum_{k=1}^{N} \int_k^{k+1} \frac{dx}{x^\alpha},$$

we get (2) in a similar manner. Note that the constants entering into the symbol $O(N^{1-\alpha})$ depend on α.

Let us now estimate the remainder R_N^α $(\alpha > 1)$:

$$\frac{1}{(\alpha-1)N^{\alpha-1}} = \int_N^{\infty} \frac{dx}{x^\alpha} = \sum_{k=N}^{\infty} \int_k^{k+1} \frac{dx}{x^\alpha} \leqslant \sum_{k=N}^{\infty} \frac{1}{k^\alpha} = R_N^\alpha;$$

$$\frac{1}{(\alpha-1)N^{\alpha-1}} \geqslant \sum_{k=N}^{\infty} \frac{1}{(k+1)^\alpha} = \sum_{k=N+1}^{\infty} \frac{1}{k^\alpha} = R_N^\alpha - \frac{1}{N^\alpha},$$

$$R_N^\alpha \leqslant \frac{1}{(\alpha-1)N^{\alpha-1}} + \frac{1}{N^\alpha} \leqslant \frac{\alpha}{\alpha-1} N^{1-\alpha} \quad (N \geqslant 1),$$

i.e. property (3) takes place.

Note that in fact we have proved more than this:

$$\frac{1}{(\alpha-1)N^{\alpha-1}} \leqslant R_N^\alpha \leqslant \frac{\alpha}{\alpha-1} \cdot \frac{1}{N^{\alpha-1}}.$$

In Problems 377 to 380 investigate the given series applying comparison tests and a necessary test:

377. $\frac{2}{3} + \frac{3}{5} + \ldots + \frac{n+1}{2n+1} + \ldots$.

378. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+2n}}$.

379. (a) $\sum_{n=1}^{\infty} \sqrt{\frac{n}{n^4+1}}$; (b) $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$.

380. (a) $\sum_{k=1}^{\infty} \ln\left(1+\frac{1}{k}\right)$; (b) $\sum_{k=1}^{\infty} \ln\left(1+\frac{1}{k^2}\right)$;

(c) $\sum_{k=1}^{\infty} \left(1-\exp\left(-\frac{1}{k^2}\right)\right)$.

In Problems 381 to 384 investigate the indicated series for convergence with the aid of D'Alembert's test or Cauchy's test.

381. $\sum_{k=1}^{\infty} \frac{2k-1}{(\sqrt{2})^k}$. **382.** $\frac{1}{3}+\frac{4}{9}+\ldots+\frac{n^2}{3^n}+\ldots \equiv \sum_{n=1}^{\infty} \frac{n^2}{3^n}$.

383. $\sum_{k=1}^{\infty} \left(\frac{k+1}{2k+1}\right)^k$. **384.** $\sum_{n=1}^{\infty} \left(\frac{n}{3n-1}\right)^{2n-1}$.

In Problems 385 to 387 investigate the given series with the aid of the integral test.

385. $\sum_{n=2}^{\infty} \frac{1}{n \ln n \cdot \ln\ln n}$. **386.** $\sum_{k=2}^{\infty} \frac{1}{k \ln^{\varepsilon} k}$ $(\varepsilon > 0)$.

387. $\sum_{n=1}^{\infty} \frac{1}{n^2+2}$.

388. Investigate for convergence the series with the general term:

(a) $u_n = \int_0^{1/n} \frac{\sqrt{x}\,dx}{x^2+1}$; (b) $u_n = \int_0^{1/n} \frac{\sin^3 x\,dx}{1+x^4}$.

In Problems 389 and 390 investigate the given series for absolute and conditional convergence.

389. $1-\frac{1}{3}+\frac{1}{5}-\ldots+\frac{(-1)^{n+1}}{2n-1}+\ldots$. **390.** $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$.

391. Show that the series $\sum_{k=1}^{\infty} a_k b_k$ is absolutely convergent if the series $\sum_{k=1}^{\infty} a_k^2$, $\sum_{k=1}^{\infty} b_k^2$ are convergent.

Sec. 5.2. Functional Series (See [1], Secs. 9.8 and 9.9)

392. Find the domain of convergence of the indicated series:

(a) $\sum_{n=1}^{\infty} \frac{1}{n^x}$; (b) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^x}$;

(c) $\sum_{k=1}^{\infty} \frac{\sin kx}{k^2}$; (d) $\sum_{n=1}^{\infty} x^{n-2}$.

393. Investigate the given sequences for uniform convergence:

(a) $f_n(x) = \frac{1}{x+n}$, $0 < x < \infty$;

(b) $f_n(x) = x^n$, $0 \leqslant x \leqslant \frac{1}{2}$;

(c) $f_n(x) = x^n - x^{n+1}$, $0 \leqslant x \leqslant 1$;

(d) $f_n(x) = x^n - x^{2n}$, $0 \leqslant x \leqslant 1$.

394. Make sure that the given series are uniformly convergent throughout the x-axis:

(a) $\sum_{k=0}^{\infty} \frac{\sin kx}{k!}$; (b) $\sum_{k=1}^{\infty} \frac{\cos kx}{2^k}$; (c) $\sum_{n=1}^{\infty} \frac{1}{n^2[1+(nx)^2]}$.

395. Applying termwise differentiation and integration, find the sum of the indicated series:

(a) $x + \frac{x^2}{2} + \ldots + \frac{x^n}{n} + \ldots$;

(b) $1 + 2x + \ldots + (n+1)x^n + \ldots$;

(c) $1 - 3x^2 + 5x^4 - \ldots + (-1)^{n-1}(2n-1)x^{2n-2} + \ldots$.

Sec. 5.3. Power Series (See [1], Secs. 9.11 and 9.12)

396. Determine the radius and the interval of convergence of the given series and investigate the convergence at the boundary points of the interval of convergence;

(a) $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$; (b) $\sum_{n=1}^{\infty} (nx)^n$; (c) $\sum_{n=2}^{\infty} (n-1)3^{n-1} x^{n-1}$.

397. Write the first two nonzero terms of the expansion into a series in powers of x for the function:

(a) $\tan x$; (b) $\tanh x$; (c) $\exp(\cos x)$.

398. Express in the form of series the following integrals:

(a) $\int_0^x \exp(-t^2)\,dt$; (b) $\int_0^x \frac{\arctan t\,dt}{t}$.

399. Evaluate to within 0.001 the integral

$$\int_{0.1}^{0.2} e^{-x}\frac{dx}{x^3}.$$

400. Expand the function e^x in powers of $(x\pm 2)$.

Chapter 6

DIFFERENTIAL EQUATIONS

(See [3], Chapter 1)

Sec. 6.1. General Concepts (See [3], Secs. 1.1 and 1.2)

401. Write the differential equation for the indicated family of curves:

(a) $y^2 = 2Cx$; (b) $y = C_1x+C_2$; (c) $y = Ce^x$;

(d) $x^2+y^2 = C^2$; (e) $y = C_1e^{2x}+C_2e^{-x}$.

402. Construct the isoclinic lines of the given differential equations and draft the integral curves:

(a) $y' = x$; (b) $y' = 1+y^2$; (c) $y' = -x$.

Sec. 6.2. First-order Equations (See [3], Sec. 1.3)

In Problems 403 to 408 solve the indicated equations with variables separable:

403. $xy\,dx+(x+1)\,dy = 0$.

404. $\sqrt{y^2+1}\,dx-xy\,dy = 0$.

405. $e^{-s}\left(1+\frac{ds}{dt}\right) = 1$.

406. $y'-xy^2 = 2xy$.

407. $y' = 3y^{2/3}$, $y(2) = 0$.

408. $y'\cot x+y = 2$, $y(0) = 1$.

409. Find the curves in which the point of intersection of any tangent with the axis of abscissas has the abscissa equal to 2/3 the abscissa of the point of tangency (Fig. 20).

410. A tank is filled with 100 litres of a solution containing 10 kg of salt. The tank is continuously supplied with water (5 litres per minute) which is mixed up with the solution. The mixture flows out at the same rate. How much salt will remain in the tank in one hour?

411. A body cools from 100° to 60° during 10 minutes. The ambient temperature is maintained at 20°. When will the body cool down to 25°?

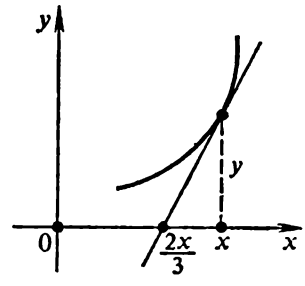

Fig. 20.

In Problems 412 to 417 solve the equations reducible to the form

$$y' = x^{\alpha-1} f\left(\frac{y}{x^{\alpha}}\right) \tag{1}$$

(for $\alpha = 1$ we have a homogeneous equation).

412. $(x+2y)\,dx - x\,dy = 0$. *Hint.* Use the substitution $y = tx$.

413. $(y^2-2xy)\,dx + x^2\,dy = 0$. **414.** $y^2 + x^2y' = xyy'$.

415. $x^4\,dy = y^2\,dx$. **416.** $x^4\,dy = \left(y^2 + \frac{9}{4}x^6\right)dx$.

417. $x\,dy = \left(x^2\cos^2\frac{y}{x^2} + 2y\right)dx$.

418. Find the curve the tangent to which is found at a distance equal to the modulus of the abscissa of the point of tangency.

An equation of the form

$$y' + a(x)y + b(x)y^2 = c(x)$$

is called the *Riccati equation*. In the general case it is not solved in quadratures. Some of Riccati equations are equations of type (1).

In Problems 419 and 420 solve the given Riccati equations:

419. $x^2y' + xy + x^2y^2 = 4$.

420. (a) $3y' = -y^2 - \frac{2}{x^2}$; (b) $y' = \sum_{i=1}^{N} a_i y_i^{\beta} x^{\alpha-1-\alpha\beta_i}$.

In Problems 421 to 426 solve the indicated linear equations:

421. $y' + 2y = 4x$. **422.** $xy' - 2y = 2x^4$.

423. $x(y'-y) = e^x$. **424.** $xy' + y = e^x$, $y(1) = 1$.

425. $y = x(y' - x\cos x)$. **426.** $(\sin^2 y + x\cot y)y' = 1$.

In Problems 427 to 430 solve the given Bernoulli equations:

427. $y' = y^4\cos x + y\tan x$. **428.** $y' = \frac{y}{x} + \frac{1}{y}$.

429. $xy' - 2x^2\sqrt{y} = 4y$. **430.** $\frac{dy}{dx} + \frac{y}{x} = -xy^2$.

Sec. 6.3. Metric Spaces. Contraction Operators. Solution Existence Theorem (See [3], Secs. 1.4-1.7)

431. Will the n-dimensional space R_n be a metric space if the distance between the points $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ is determined by the equality:

(a) $\varrho(\mathbf{x}, \mathbf{y}) = \max\limits_{i=1,\ldots,n} \{|x_i - y_i|\}$;

(b) $\varrho(\mathbf{x}, \mathbf{y}) = \begin{cases} 1, & \mathbf{x} \neq \mathbf{y}, \\ 0, & \mathbf{x} = \mathbf{y}? \end{cases}$

432. Find out whether the set of all continuous functions defined on $[a, b]$ is a metric space if

$$\varrho(f, g) = \left(\int_a^b [f(x) - g(x)]^2 \, dx \right)^{1/2}.$$

433. Let

$$f_n(x) = \begin{cases} n^\alpha, & 0 \leqslant x < 1/n, \quad \alpha > 0, \\ 0, & 1/n \leqslant x \leqslant 1. \end{cases}$$

For what α does the sequence $f_n(x)$ converge to zero in the sense of the metric from Problem 432?

434. Will the metric space $M = [2, 3)$ with the metric $\varrho(x, y) = |x-y|$ be a complete metric space?

435. Will the function $F(x) = x^2$ be a contraction operator:

(a) in the complete metric space $M = [-1/3, 1/3]$;

(b) in the complete metric space $M = [-1, 1]$?

In both cases the metric is $\varrho(x, y) = |x-y|$.

436. Construct an iterated sequence for the operator $F(x) = x^2$ if $x_0 = 1/2$.

437. (a) Find the fixed points of the operator $F(x) = 1/(1+x)$ on $[1/2, 1]$. Will the operator $F(x)$ be contracting on $[1/2, 1]$?

(b) Let the operator $F(x) (x = (x_1, x_2))$ operate in the two-dimensional metric space R_2 according to the law

$$F(x) = (x_1, -x_2)$$

(mirror image about the axis $x_2 = 0$). What points are fixed for this operator?

(c) Let $F(x) = (x_1, x_2^2)$, $x \in R_2$. What points of the plane R_2 are the fixed points of the operator F?

438. On the basis of the theorem on existence of a solution of a differential equation, investigate in what interval $[x_0-\delta, x_0+\delta]$ the existence of a solution of the equation $y' = f(x, y)$ is guaranteed if

(a) $x_0 = 1$, $y_0 = y(1) = 2$, $f(x, y) = 2xy^2$ on the set

$$D = \left\{ \begin{matrix} |x-1| \leqslant 1 = a \\ |y-2| \leqslant 1 = b \end{matrix} \right\};$$

(b) $x_0 = 0$, $y_0 = y(0) = 1$, $f(x, y) = 2xy^2$ on the set

$$D = \left\{ \begin{array}{l} |x| \leqslant \sqrt{2}/4 = a \\ |y-1| \leqslant 1 = b \end{array} \right\}.$$

439. For the equation $y' = \frac{1}{2}xy$, $y(0) = 1$, evaluate approximately $y(1)$ applying Euler's method. Take $h = 0.1$ for the computation step.

440. For the equation $y' = x+y$ evaluate approximately $y(2)$ if $y(1) = 1$. Take $h = 0.1$ for the computation step.

Sec. 6.4. Equations Not Resolved with Respect to Derivative. Singular Solutions (See [3], Secs. 1.8-1.10)

441. Find all the solutions of the given equations; single out singular solutions if any; give a pictorial representation for each case:

(a) $y'^2-y^2 = 0$; (b) $y'^2-4y^3 = 0$;

(c) $y^2(y'^2+1) = 1$; (d) $y'^2 = 4y^3(1-y)$;

(e) $xy'^2-2yy'+x = 0$; (f) $y(xy'-y)^2 = y-2xy'$.

442. Solve the given equations by introducing a parameter:

(a) $x = y'^3+y'$; (b) $y = y'^2+2y'^3$; (c) $x = y'\sqrt{1+y'^2}$.

443. An equation of the form

$$y = x\varphi(y')+\psi(y'),$$

where φ, ψ are some functions, is called *Lagrange's differential equation*. In particular, if $\varphi(y') \equiv y'$, then it is called *Clairaut's differential equation*. These equations are also solved by introducing a parameter:

$$y' = p, \quad dy = p\,dx; \quad y = x\varphi(p)+\psi(p);$$
$$dy = \varphi(p)\,dx+x\varphi'(p)\,dp+\psi'(p)\,dp,$$

or, taking into account that $dy = p\,dx$, we get

$$[p-\varphi(p)]\,dx = [x\varphi'(p)+\psi'(p)]\,dp.$$

The latter equation is linear with respect to x. We know how to solve it (if $p \not\equiv \varphi(p)$): $x = Cf(p)+g(p)$, where f, g are known functions. The system

$$\left\{ \begin{array}{l} x = Cf(p)+g(p), \\ y = x\varphi(p)+\psi(p) \end{array} \right.$$

yields a parametric representation of the solution.

And if $p \equiv q(p)$ (in this case we have a Clairaut's equation), then

$$[x\varphi'(p)+\psi'(p)]\,dp = 0,$$

whence:

(1) $dp = 0$, $p = C$ and $y = x\varphi(C)+\psi(C) = xC+\psi(C)$ is the general solution of a Lagrange's (Clairaut's) equation. This is a family of straight lines. Formally, the general solution is obtained by replacing y' by an arbitrary constant C;

(2) $x\varphi'(p)+\psi'(p) = 0$. Then from the system

$$\begin{cases} y = x\varphi(p)+\psi(p), \\ 0 = x\varphi'(p)+\psi'(p) \end{cases}$$

by eliminating the parameter p, we obtain $y = \chi(x)$. If this function is a solution of Lagrange's equation and the uniqueness of solution is violated, then it is a singular solution of Lagrange's (Clairaut's) equation.

444. Solve the Clairaut's equation

$$y = xy' - \frac{1}{4}y'^2.$$

445. Solve the equation

$$y = xy' + \sqrt{1+y'^2}.$$

Sec. 6.5. Reducing the Order of a Differential Equation (See [3], Sec. 1.14)

446. Solve the given equations: (a) $y'' = \cos x$; (b) $y''' = x$. In Problems 447 to 457 solve the given equations.

447. $x^2y'' = y'^2$.

448. $y''' = 2(y''-1)\cot x$.

449. $y''' = y''$.

450. $y'' = 2yy'$.

451. $xy'' = 2yy'-y'$.

452. $yy''+y'^2 = 1$.

453. $2yy'' = y'^2+y^2$.

454. $yy'' = (y')^3$.

455. $yy'' = y'^2+y^2y'$.

456. $xyy''-xy'^2-yy' = 0$.

457. $x^2yy'' = (y-xy')^2$.

Sec. 6.6. Linear Equations with Constant Coefficients (See [3], Sec. 1.16)

In Problems 458 to 462 solve the following equations:

458. $y''-2y'-3y = 0$.

459. $y'''-5y''+6y' = 0$.

460. $y^{(4)}-y = 0$.

461. $y'''-5y''-8y'-4y = 0$.

462. (a) $y^{(4)}+2y''+y=0$; (b) $y''+3y'-4y=0$.

463. Prove that the Wronskian

$$W(t)\equiv W[y_1,\ \ldots,\ y_n]=\begin{vmatrix} y_1(t) & \ldots & y_n(t) \\ y_1'(t) & \ldots & y_n'(t) \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ y_1^{(n-1)}(t) & \ldots & y_n^{(n-1)}(t) \end{vmatrix}$$

of the system of solutions $y_1(t),\ \ldots,\ y_n(t)$ of the equation

$$y^{(n)}(t)+p_1(t)\,y^{(n-1)}(t)+\ \ldots\ +p_n(t)\,y(t)=0,$$

with continuous on $(a,\ b)$ coefficients $p_1(t),\ \ldots,\ p_n(t)$ satisfies the equation

$$\frac{dW(t)}{dt}=-p_1(t)\,W(t).$$

The given equation is an equation with variables separable, therefore, integrating it within the limits from x_0 to x, we obtain

$$\ln\left|W(t)\right|_{t=x_0}^{x}=-\int_{x_0}^{x}p_1(t)\,dt,$$

$$W(x)=W(x_0)\exp\left(-\int_{x_0}^{x}p_1(t)\,dt\right).$$

The last formula is known as the *Ostrogradski-Liouville formula.* It implies, among other things, that if $W(x_0)\neq 0$, then the Wronskian is not equal to zero at all points belonging to $(a,\ b)$. In this case, as we know, the solutions $y_1(t),\ \ldots\ y_n(t)$ are linearly independent on $(a,\ b)$.

Hint. See Problem 86, (b). Take advantage of the fact that

$$y_k^{(n)}(t)=-p_1(t)\,y_k^{(n-1)}(t)-\ \ldots\ -p_n(t)\,y_k(t)\qquad (k=1,\ \ldots,\ n).$$

464. Solve the following nonhomogeneous equations

$$y'''-5y''+8y'-4y=f(x),\quad \text{where}$$

(a) $f(x)=2e^{3x}$; (b) $f(x)=4e^{x}$; (c) $f(x)=3e^{2x}$.

465. Find the particular solution of the following equations

$$y^{(4)}+2y''+y=f(x),\quad \text{where}$$

(a) $f(x)=e^{x}$; (b) $f(x)=\sin x$;

(c) $f(x)=\sin x+\cos x$; (d) $f(x)=\sin 2x$.

466. Solve the equation $y''+2y'-3y = \sin x$.

467. Write the form of the particular solution of the indicated nonhomogeneous equations:

$$y'''-5y''+6y' = f(x),$$

where

(a) $f(x) = e^x$; (b) $f(x) = xe^{2x}$; (c) $f(x) = \sin x$;

(d) $f(x) = x^4e^{3x}$; (e) $f(x) = x^2+x+1$; (f) $f(x) = xe^x \sin x$.

Sec. 6.7. Euler's Equation. Equations with Variable Coefficients (See [3], Secs. 1.15 and 1.16)

468. Find out which of the given systems of functions are linearly independent on [0, 1]:

(a) 1, $\sin^2 x$, $\cos 2x$; (b) $4-x$, $2x+3$, $6x+8$;

(c) $x+2$, $(x+2)^2$; (d) x^2, x^3, x^4.

469. Solve Euler's equations:

(a) $x^2y''-4xy'+6y = 0$; (b) $x^2y''-3xy'+3y = 0$;

(c) $x^2y''-3xy'+4y = 0$; (d) $x^2y'''-2y' = 0$.

470. Solve the given equation (a particular case of Bessel's differential equation for $\nu = 1/4$, see [3], Sec. 1.24)

$$x^2y''+xy'+\left(x^2-\frac{1}{4}\right)y = 0.$$

Solution. Let us introduce the substitution $y = \alpha(x)z$ and choose the function $\alpha(x)$ so as to eliminate the term containing the first derivative z'. We have

$$y' = \alpha'z+\alpha z', \quad y'' = \alpha''z+2\alpha'z'+\alpha z'';$$

$$x^2\alpha z''+z'[2\alpha'x^2+\alpha x]+z\left[\alpha''x^2+\alpha'x+\alpha x^2-\frac{\alpha}{4}\right] = 0;$$

$$2\alpha'+\frac{\alpha}{x} = 0, \quad \alpha = \frac{1}{\sqrt{x}}, \quad \alpha' = -\frac{1}{2x\sqrt{x}}, \quad \alpha'' = \frac{3}{4x^2\sqrt{x}},$$

$$x^2\alpha''+x\alpha'+x^2\alpha-\frac{\alpha}{4} = \frac{x^2}{\sqrt{x}}.$$

We finally obtain

$$\frac{x^2}{\sqrt{x}}z''+\frac{x^2}{\sqrt{x}}z = 0, \quad z''+z = 0.$$

The last equation is with constant coefficients, its general solution being

$$z = C_1 \cos x + C_2 \sin x.$$

The general solution of the original equation is

$$y = \frac{1}{\sqrt{x}}(C_1 \cos x + C_2 \sin x).$$

471. Solve the equation

$$y'' - 2xy' + x^2 y = 0.$$

Hint. See Problem 470: $\alpha(x) = \exp(x^2/2)$.

472. Solve the equation $x^2y'' + xy' - y = f(x)$, where

(a) $f(x) = x^2$; (b) $f(x) = x^{10}$.

Sec. 6.8. The Method of Variation of Constants (See [3], Sec. 1.17)

473. Solve the following equations:

(a) $y'' + y = \frac{1}{\sin x}$, $y\left(\frac{\pi}{2}\right) = 1$, $y'\left(\frac{\pi}{2}\right) = 0$;

(b) $y'' + 4y = 2 \tan x$; (c) $y''' + y' = \frac{1}{\cos x}$;

(d) $y'' + 2y' + y = \frac{1}{x} e^{-x}$.

Sec. 6.9. Systems of Differential Equations (See [3], Secs. 1.19-1.22)

474. Solve the given system by reducing it to one differential equation:

(a) $\begin{cases} \frac{dy}{dx} + 2y + 4z = 4x, \\ \frac{dz}{dx} + y - z = x^2; \end{cases}$ (b) $\begin{cases} \frac{dy}{dx} = z, \\ \frac{dz}{dx} = -y; \end{cases}$

(c) $\begin{cases} \frac{dx}{dt} - y - z = 0, \\ -x + \frac{dy}{dt} - z = 0, \\ -x - y + \frac{dz}{dt} = 0; \end{cases}$ (d) $\begin{cases} \frac{dy}{dx} - z = 1. \\ \frac{dz}{dx} + y = x. \end{cases}$

475. Solve the following homogeneous systems without passing to one differential equation:

(a) $\begin{cases} \dot{y}_1 = 2y_1+y_2, \\ \dot{y}_2 = 3y_1+4y_2; \end{cases}$ (b) $\begin{cases} \dot{x} = x-y, \\ \dot{y} = y-4x; \end{cases}$

(c) $\begin{cases} \dfrac{dy}{dx} = y+5z, \\ \dfrac{dz}{dx} = -y-3z; \end{cases}$ (d) $\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = z(t), \\ \dot{z}(t) = x(t). \end{cases}$

476. Solve the following nonhomogeneous systems:

(a) $\begin{cases} \dot{x} = x-y+1, \\ \dot{y} = y-4x+t; \end{cases}$ (b) $\begin{cases} \dot{x} = x-y+e^t, \\ \dot{y} = x-4y+e^{3t}. \end{cases}$

Sec. 6.10. Solving Equations with the Aid of Power Series (See [3], Sec. 1.24)

477. $y''+y'-xy^2 = 0, \quad y(0) = 2, \quad y'(0) = 1.$

478. $y''+xy = 0, \quad y(0) = 1, \quad y'(0) = 0.$

Solution. We shall seek for the solution of $y(x)$ in the form of the power series

$$y(x) = a_0+a_1x+a_2x^2+ \ldots .$$

Finding y', y'', xy and substituting them into the equation, we obtain a number of relations connecting the coefficients a_i ($a_0 = 1$, $a_1 = 0$) wherefrom we just find the the values of these coefficients.

But we may also reason in the following way. Since the power series is at the same time the Taylor series of the function $y(x)$, me may write

$$y(x) = y(0)+y'(0)x+\frac{y''(0)}{2!}x^2+\frac{y'''(0)}{3!}x^3+ \ldots .$$

The values $y(0) = 1, y'(0) = 0$ are known by hypothesis. The values of other derivatives obtained from the solution of $y(x)$ at the point $x = 0$ can be found with the aid of the differential equation. From the equation we have $y''(0) = -0\cdot y(0) = 0$. Differentiating the equation, we obtain

$$y'''+y+xy' = 0,$$

whence

$$y'''(0) = -y(0) = -1.$$

Analogously,

$$y^{(4)}+2y'+xy'' = 0, \qquad y^{(4)}(0) = -2y'(0) = 0,$$

$$y^{(5)}+3y''+xy''' = 0, \qquad y^{(5)}(0) = -3y''(0) = 0,$$

$$y^{(6)}+4y'''+xy^{(4)}=0, \qquad y^{(6)}(0)=-4y'''(0)=4,$$

. . . . . .

$$y^{(n)}+(n-2)y^{(n-3)}+xy^{(n-2)}=0, \quad y^{(n)}(0)=-(n-2)\,y^{(n-3)}(0),$$

. . . . . .

Substituting the value $y^{(n)}(0)$, we get

$$y(x)=1-\frac{x^3}{3!}+\frac{1\cdot 4}{6!}x^6-\frac{1\cdot 4\cdot 7}{9!}x^9+\ \ldots\ .$$

The obtained series converges throughout the axis uniformly and absolutely.

479. $y''+ye^x=0, \quad y(0)=2, \quad y'(0)=1.$

480. $y'=y+xe^y, y(0)=0$. Find the first three terms of the series.

Sec. 6.11. Stability in the Sense of Lyapunov (See [3], Secs. 1.25 and 1.26)

481. Investigate the trivial solution of the indicated system for stability with the aid of Lyapunov function:

(a) $\begin{cases} \dot{x}=-x^3-y, \\ \dot{y}=x-y^3; \end{cases}$ (b) $\begin{cases} \dot{x}=2y^3-x^5, \\ \dot{y}=-x-y^3+y^5; \end{cases}$

(c) $\begin{cases} \dot{x}=x^3-y, \\ \dot{y}=x+y^3. \end{cases}$

Remark. If there exists a function $v(x, y)$ such that in a sufficiently small neighbourhood of the origin there is a region where $v>0$ and $v=0$ on a part of the boundary of the region ($v>0$) and $\frac{dv}{dt}=\frac{\partial v}{\partial x}\frac{dx}{dt}+\frac{\partial v}{\partial y}\frac{dy}{dt}>0$ in the region $v>0$, then the stationary point is unstable (Cetaev's Instability theorem).

482. Investigate the trivial solution for stability in the following systems with a symmetric matrix:

(a) $\begin{cases} \dot{x}=-2x+y, \\ \dot{y}=x-y; \end{cases}$ (b) $\begin{cases} \dot{x}=x-2y, \\ \dot{y}=-2x+4y; \end{cases}$ (c) $\begin{cases} \dot{x}=x+3y, \\ \dot{y}=3x+8y. \end{cases}$

483. Investigate the given systems for stability:

(a) $\begin{cases} \dot{x}=3x, \\ \dot{y}=2x+y; \end{cases}$ (b) $\begin{cases} \dot{x}=x+3y, \\ \dot{y}=-6x-5y; \end{cases}$ (c) $\begin{cases} \dot{x}=x, \\ \dot{y}=2x-y; \end{cases}$

(d) $\begin{cases} \dot{x}=-2x-5y, \\ \dot{y}=2x+2y; \end{cases}$ (e) $\begin{cases} \dot{x}=x+y, \\ \dot{y}=y. \end{cases}$

Chapter 7

MULTIPLE INTEGRALS
(See [3], Chapter 2)

Sec. 7.1. Integrals Depending on a Parameter (See [3], Sec. 2.4)

484. Find the domain of definition of the function

$$F(x) = \int_{\pi}^{2\pi} \frac{\sin xy}{y}\, dy$$

(i.e. the set of values of x where the integral exists).

Investigate the given function for continuity and differentiability.

485. Proceeding from the equality

$$\int_0^b \frac{dx}{x^2+a^2} = \frac{1}{a} \operatorname{arc\,tan} \frac{b}{a},$$

evaluate the integral

$$\int_0^b \frac{dx}{(x^2+a^2)^2}.$$

486. Compute the derivative $F'(x)$ if

(a) $F(x) = \int_x^{x^2} e^{-xy^2}\, dy$; (b) $F(x) = \int_0^x \frac{\ln(1+xy)}{y}\, dy$ $(x > 0)$;

(c) $F(x) = \int_0^x f(y+x,\ y-x)\, dy,$

where $f(u, v)$ is continuous together with its partial derivative f'_u.

487. Find the integral of the function $F(x)$ on $[0, 1]$ if

(a) $F(x) = \int_{x^2}^1 (y-2x)\, dy$; (b) $F(x) = \int_x^1 (x+y)\, dy.$

Sec. 7.2. Multiple Integrals (See [3], Secs. 2.1-2.5)

In Problems 488 to 490 compute the given double integrals.

488. $\iint\limits_D \frac{dx\,dy}{(x+y)^2}$, where $D = \{3 \leqslant x \leqslant 4,\ 1 \leqslant y \leqslant 2\}$.

489. $\iint\limits_D (x+2y)\,dx\,dy$, where $D = \left\{\begin{matrix} y^2-4 \leqslant x \leqslant 5 \\ -3 \leqslant y \leqslant 3 \end{matrix}\right\}$.

490. $\iint\limits_D r^2 \sin^2\varphi\,dr\,d\varphi$, where $D = \left\{\begin{matrix} 0 \leqslant r \leqslant 3\cos\varphi \\ -\pi/2 \leqslant \varphi \leqslant \pi/2 \end{matrix}\right\}$.

In Problems 491 to 495 draw the range of integration and change the order of integration of the given double integrals.

491. $I = \int\limits_{-6}^{2} dy \int\limits_{(y^2-4)/4}^{2-y} f(x, y)\,dx$.

492. $I = \int\limits_{0}^{3} dx \int\limits_{x/3}^{2x} f(x, y)\,dy$.

493. $I = \int\limits_{1}^{2} dx \int\limits_{1/x}^{x} f(x, y)\,dy$. **494.** $I = \int\limits_{0}^{1} dx \int\limits_{x^4}^{x^2} f(x, y)\,dy$.

495. $I = \int\limits_{0}^{c\sqrt{2}} dy \int\limits_{y^2/2c^2}^{\sqrt{3-\frac{y^2}{c^2}}} f(x, y)\,dx$.

In Problems 496 and 497 change the order of integration and evaluate the double integral.

496. $I = \int\limits_{0}^{1} dx \int\limits_{0}^{x} (x+y^2)\,dy + \int\limits_{1}^{2} dx \int\limits_{0}^{2-x} (x+y^2)\,dy$.

497. $I = \int\limits_{0}^{1} dx \int\limits_{x^2}^{x} xy^2\,dy$.

In Problems 498 to 500 evaluate the given triple integrals.

498. $\int\limits_{0}^{a} dx \int\limits_{0}^{b} dy \int\limits_{0}^{c} (x+y+z)\,dz$. **499.** $\int\limits_{0}^{1} dx \int\limits_{0}^{x} dy \int\limits_{0}^{y} xyz\,dz$.

500. $\int\limits_0^4 dz \iint\limits_D xyz\,dx\,dy, \quad D = \{x^2+y^2 \leqslant 4, \quad x > 0, \quad y > 0\}.$

501. Set the limits of integration in the triple integral

$$\iiint\limits_V f(x, y, z)\,dx\,dy\,dz,$$

where

(a) V is a common part of the paraboloid $2az \geqslant x^2+y^2$ and the sphere $x^2+y^2+z^2 \leqslant 3a^2 \quad (a > 0)$.

(b) V is a common part of the spheres $x^2+y^2+z^2 \leqslant R^2$, $x^2+y^2+z^2 \leqslant 2Rz$.

Sec. 7.3. Change of Variables in a Multiple Integral (See [3], Secs. 2.6-2.10)

502. Passing to polar coordinates, evaluate the double integral

$$\iint\limits_S \sqrt{a^2-x^2-y^2}\,dx\,dy,$$

where $S = \{x^2+y^2 \leqslant a^2, \ y > 0\}$.

503. Compute the triple integral

$$I = \iiint\limits_V \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right) dx\,dy\,dz,$$

where V is an ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \leqslant 1$.

504. Compute the double integral

$$\int\limits_0^a dx \int\limits_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\,dy.$$

505. Compute the double integral

$$\iint\limits_S \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dx\,dy,$$

where S is an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} \leqslant 1$.

506. Passing to spherical coordinates, evaluate

$$\iiint\limits_V \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz,$$

where V is a sphere of radius R with centre at the origin.

507. Passing to cylindrical coordinates, evaluate the integral

$$I = \int_0^2 dx \int_0^{\sqrt{2x-x^2}} dy \int_0^a z\sqrt{x^2+y^2}\, dz.$$

508. Compute the double integral

$$\iint_S (x^2+y^2)\sqrt{R^2-x^2-y^2}\, dx\, dy,$$

where S is a circle of radius R with centre at the origin.

Sec. 7.4. Application of Multiple Integrals (See [3], Secs. 2.11 and 2.12)

509. Compute the area of the figure bounded by the parabolas $y^2 = 10x+25$ and $y^2 = -6x+9$ and make the relevant drawing.

510. Draw the body whose volume is expressed by the double integral

$$\int_0^1 dx \int_0^{1-x} (1-x-y)\, dy,$$

and compute its volume V.

511. Find the volume V of the solid bounded by the cylinder $x^2+z^2 = a^2$ and the planes $y = 0$, $z = 0$, $y = x$. Make the relevant drawing.

512. Compute the volume V of the part of the cylinder $x^2+y^2 = 2ax$ enclosed between the paraboloid $x^2+y^2 = 2az$ and the plane $z = 0$. Make the relevant drawing.

513. Find the area of the portion of the plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1,$$

located in the first octant ($x > 0$, $y > 0$, $z > 0$).

514. Find th surface area of the part of the sphere $x^2+y^2+z^2 = a^2$ cut away by the elliptical cylinder

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \quad (b < a).$$

515. Find the centre of gravity of the upper half of the ellipse

$$D = \left\{\frac{x^2}{a^2}+\frac{y^2}{b^2} \leqslant 1, \quad y > 0\right\},$$

filled with mass of density $\varrho \equiv 1$.

516. Find the volume:

(a) of the solid generated by revolving a half of the ellipse D (see Problem 515) about the x-axis;

(b) of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \leqslant 1$.

517. In the hemisphere

$$\Omega = \{x^2+y^2+z^2 \leqslant a^2, \quad z \geqslant 0\}$$

the density of distribution of masses is proportional to the distance of a point from the centre of the sphere: $\varrho(x, y, z) = c\sqrt{x^2+y^2+z^2}$. Find the centre of gravity of this solid.

518. Find the centre of gravity of the homogeneous figure shown in Fig. 21:

$$S = \{0 \leqslant y \leqslant 4-x^2, \quad -2 \leqslant x \leqslant 2\}.$$

519. Find the volume of the solid generated by rotating the curvilinear trapezoid S (see the preceding problem) about the x-axis.

520. Find the moment of inertia of a homogeneous cylinder (whose altitude is h and the radius of the base is a) about the axis which serves as the diameter of the base of the cylinder.

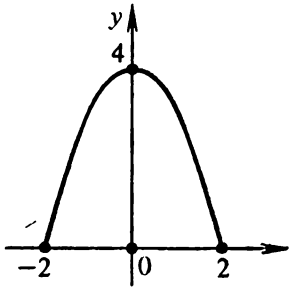

Fig. 21.

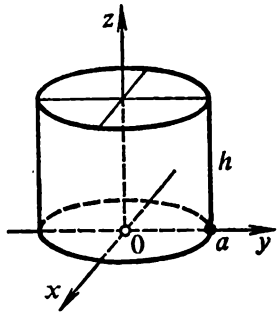

Fig. 22.

Solution. Let the z-axis be directed along the axis of the cylinder, the base of the cylinder be found in the plane $z = 0$, and let the centre of the base coincide with the origin. We shall seek for the moment of inertia about the y-axis (i.e. about the planes $x = 0$, $z = 0$) (Fig. 22):

$$I_{x,z}^{(2)} = \int_G (x^2+z^2)\, dx\, dy\, dz,$$

where G is the cylinder under consideration. Computing the integral, we obtain

$$I^{(2)}_{(x,z)} = \int_{-a}^{a} dx \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy \int_0^h (x^2+z^2)\,dz =$$

$$= 4h\int_0^a \sqrt{a^2-x^2}\left(x^2+\frac{h^2}{3}\right)dx = (x = a\sin t) =$$

$$= 4a^2h\int_0^{\pi/2} \cos^2 t\left(a^2\sin^2 t+\frac{h^2}{3}\right)dt =$$

$$= 4a^2h\left\{a^2\int_0^{\pi/2}\frac{1-\cos^2 2t}{4}\,dt+\frac{h^2}{3}\int_0^{\pi/2}\frac{1+\cos 2t}{2}\,dt\right\} = \frac{a^2h\pi}{12}(3a^2+4h^2).$$

Sec. 7.5. Improper Integrals (See [3], Sec. 2.13)

521. Compute the following integrals:

(a) $\int_0^\infty \int_0^\infty \frac{dx\,dy}{(1+x^2+y^2)^2}$; (b) $\int_0^\infty \int_0^\infty \frac{dx\,dy}{1+x^2+y^2}$;

(c) $\int_0^\infty \int_0^\infty xy\exp(-x^2-y^2)\,dx\,dy.$

In Problems 522 and 523 compute the given integrals by differentiation with respect to a parameter:

522. $\int_0^\infty \frac{1-e^{-xy}}{xe^x}\,dx = F(y) \quad (y > -1).$

523. $\int_0^\infty \frac{1-e^{-yx^2}}{x^2}\,dx = F(y) \quad (y > 0).$

524. Using the equality

$$\int_0^1 x^y\,dx = \frac{1}{y+1} \quad (y > -1),$$

evaluate the integral

$$\int_0^1 \frac{x^\alpha - x^\beta}{\ln x}\, dx \quad (\beta > -1, \quad \alpha > -1).$$

525. Investigate for convergence the improper integral

$$\iint_S \ln \sqrt{x^2+y^2}\, dx\, dy,$$

where S is a circle $x^2+y^2 \leqslant 1$.

526. Investigate the indicated integrals for uniform convergence:

(a) $F(y) = \int_0^\infty \frac{\cos xy}{1+x^2}\, dx \quad (-\infty < y < \infty)$;

(b) $F(y) = \int_0^\infty \sqrt{y} \exp(-yx^2)\, dx \quad (0 \leqslant y < \infty)$.

Chapter 8

VECTOR ANALYSIS
(See [3], Chapter 3)

Sec. 8.1. Line Integrals of the First Kind (See [3], Sec. 3.2)

In Problems 527 to 532 evaluate the given line integrals:

527. $\int_\Gamma \frac{ds}{x-y}$, where Γ is a segment of the straight line connecting the points $A = (0, -2)$ and $B = (4, 0)$.

528. $\int_\Gamma xy\,ds$, where Γ is the contour of the triangle with vertices $A = (-1, 0)$, $B = (1, 0)$, $C = (0, 1)$.

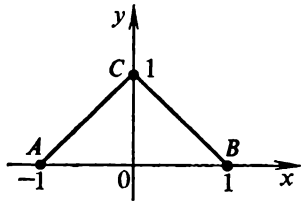

Fig. 23.

Solution. The equation of the straight line AB: $y = 0$; of the straight line BC: $x+y = 1$; of the straight line AC: $y-x = 1$ (Fig. 23).

$$\int_{AB} xy\,ds = 0;$$

$$\int_{BC} xy\,ds = \int_{CB} xy\,ds = \int_0^1 x(1-x)\sqrt{2}\,dx = \sqrt{2}/6;$$

$$\int_{CA} xy\,ds = \int_{AC} xy\,ds = \int_{-1}^0 x(1+x)\sqrt{2}\,dx = -\sqrt{2}/6;$$

$$\int_\Gamma xy\,ds = \int_{AB} \ldots + \int_{BC} \ldots + \int_{CA} \ldots = 0 + \frac{\sqrt{2}}{6} - \frac{\sqrt{2}}{6} = 0.$$

529. $\int_\Gamma xy\,ds$, where Γ is the contour of the rectangle with vertices $A = (0, 0)$, $B = (4, 0)$, $C = (4, 2)$, $D = (0, 2)$.

530. $\int_\Gamma xy\,ds$, where Γ is the part of an ellipse found in the first quadrant: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $x \geqslant 0$, $y \geqslant 0$.

531. $\int\limits_{\Gamma} y\,ds$, where Γ is the part of the parabola $y = 2\sqrt{x}$, situated in the upper half-plane $0 \leqslant x \leqslant 1$.

532. $\int\limits_{\Gamma} (x-y)\,ds$, where Γ is the circle: $x^2+y^2 = 2ax$.

533. Find the mass m of the portion of the ellipse $x = a\cos t$, $y = b\sin t$ situated in the first quadrant if the density at each of its points is equal to the ordinate of this point.

534. Find the area S of the lateral surface of the parabolic cylinder $y = \frac{3}{8}x^2$ bounded by the planes $z = 0$, $x = 0$, $z = x$, $y = 6$.

Solution. Geometrically, the line integral

$$\int\limits_{\Gamma} f(x, y)\,ds,$$

where $f(x, y) \geqslant 0$, can be interpreted as the area of a cylindrical surface with generatrix parallel to the z-axis whose base is the contour of integration Γ and whose altitudes are equal to the values of the function $f(x, y)$. Therefore the desired area (see Fig. 24)

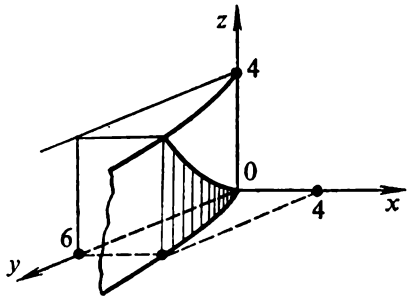

Fig. 24.

$$S = \int\limits_{\Gamma} x\,ds,$$

where Γ is part of the parabola $y = \frac{3}{8}x^2$ $(0 \leqslant x \leqslant 4)$. Computing, we obtain

$$S = \int\limits_0^4 x\sqrt{1+\left(\frac{3}{4}x\right)^2}\,dx = \frac{1}{72}\int\limits_0^4 \sqrt{16+9x^2}\,d(16+9x^2)$$

$$= \frac{1}{108}(16+9x^2)^{3/2}\Big|_0^4 = \frac{16}{27}(10\sqrt{10}-1).$$

535. Compute the area of the lateral surface of a circular cylinder found under the first turn of the helical line $x = a\cos t$, $y = a\sin t$, $z = bt$ $(0 \leqslant t \leqslant 2\pi)$ and above the plane $z = 0$.

536. Find the coordinates of the centre of gravity of a homogeneous half-arch of the cycloid $x = a(t-\sin t)$, $y = a(1-\cos t)$ $(0 \leqslant t \leqslant \pi)$.

537. Find the moment of intertia about the z-axis (that is, with respect to the planes $x = 0$, $y = 0$) of the first turn Γ of the helical line $x = a\cos t$, $y = a\sin t$, $z = bt$ $(0 \leqslant t \leqslant 2\pi)$.

Sec. 8.2. Integral of a Vector Along a Curve (See [3], Sec. 3.3)

538. Evaluate the line integral of the second kind

$$\int_{\Gamma} (y^2\,dx + x^2\,dy),$$

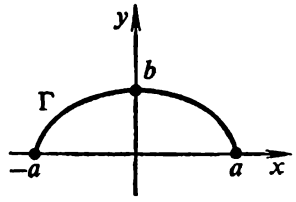

Fig. 25.

where Γ is the upper half of the ellipse $x = a\cos t$, $y = b\sin t$ traced clockwise (Fig. 25).

Solution. When a point moves along the curve Γ in the indicated direction, the parameter t changes from π to 0.

Therefore

$$\int_{\Gamma} (y^2\,dx + x^2\,dy)$$

$$= \int_{\pi}^{0} [b^2\sin^2 t(-a\sin t) + a^2\cos^2 t \cdot b\cos t]\,dt$$

$$= ab\int_{0}^{\pi} [b\sin^3 t - a\cos^3 t]\,dt$$

$$= ab\left[-\int_{0}^{\pi} b(1-\cos^2 t)d\cos t - \int_{0}^{\pi} a(1-\sin^2 t)d\sin t\right] = \frac{4}{3}ab^2.$$

539. Compute

$$\int_{\Gamma} x\,dy,$$

where Γ is the contour of the triangle formed by the coordinate axes and the straight line $x+y = 2$ traversed in the positive direction (i.e. anticlockwise).

540. Compute the value of the integral

$$\int_{\Gamma} (y^2\,dx + x^2\,dy),$$

where: (a) Γ is the arc of the parabola $y = 4 - x^2$ found in the upper half-plane and traversed clockwise; (b) Γ is the polygonal line connecting the points $(-2, 0)$, $(0, 4)$, $(2, 0)$; (c) Γ is the segment $[-2, 2]$ of the x-axis.

541. Evaluate the integral

$$\int_{\Gamma} (x\,dy + y\,dx),$$

where (a) Γ is an arc of the parabola $y = \sqrt{x}$, (b) Γ is a segment of a straight line, (c) Γ is the arc of the parabola $y = x^2$ connecting the points (0, 0) and (1, 1) in the direction indicated by the arrows (see Fig. 26).

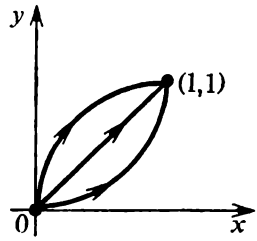

Fig. 26.

Sec. 8.3. Potential. The Curl of a Vector (See [3], Sec. 3.4)

542. Find the gradient of the function

$$u = x^2 + 2y^2 + 3z^2 + xy - 6z.$$

543. At what points in space is the gradient of the field

$$u = x^3 + y^3 + z^3 - 3xyz$$

(a) perpendicular to the z-axis? (b) equal to zero?

544. Find the curl of the vector $\mathbf{a} = \{x, y, z\}$, i.e. the radius vector, of the point (x, y, z).

545. Find the curl of the vector $\mathbf{a} = \{z + y, x, y\}$.

546. Find out whether the vectors (a) $\mathbf{a} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$, (b) $\mathbf{a} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$, (c) $\mathbf{a} = z\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ have a potential in the entire space R_3.

547. Find the potential function for the vector $\mathbf{a} = \{x^2, y^2, z^2\}$ in the space R_3.

Solution. The curl of the vector $\mathbf{a}$ is equal to zero in R_3. Besides, the space R_3 represents a simply connected domain. Therefore the vector $\mathbf{a}$ has a potential which is found by the formula

$$U(x, y, z) = \int_{\Gamma} (P\,dx + Q\,dy + R\,dz),$$

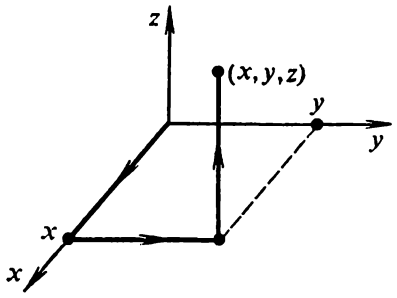

Fig. 27.

where for the curve Γ we may take the polygonal line (Fig. 27) connecting the points $(0, 0, 0)$, $(x, 0, 0)$, $(x, y, 0)$, (x, y, z). In this case the integral of the second kind is independent of the integration path. Computing, we find

$$U(x, y, z) = \frac{1}{3}(x^3+y^3+z^3).$$

Sec. 8.4. Exact Differential Equations of First Order (See [3], Sec. 3.5)

548. Which of the given equations are exact differential equations:

(a) $(2x+3y)\,dx+(3x-4y)\,dy = 0$,

(b) $\frac{dx}{y}-\frac{x\,dy}{y^2} = 0$, (c) $(2-y)\,dx+x\,dy = 0$?

In Problems 549 to 555 solve the given equations:

549. $(2x+3x^2y)\,dx+(x^3-3y^2)\,dy = 0$.

550. $2xy\,dx+(x^2-y^2)\,dy = 0$.

551. $e^{-y}\,dx-(2y+xe^{-y})\,dy = 0$.

552. $\frac{3x^2+y^2}{y^2}\,dx-\frac{2x^3+5y}{y^3}\,dy = 0$.

553. $yz\,dx+xz\,dy+xy\,dz = 0$.

554. $(x+z)\,dy+(y+z)\,dx+(x+y)\,dz = 0$.

555. $2xy\,dx+(x^2+z^2)\,dy+2yz\,dz = 0$.

Sec. 8.5. Green's Formula (See [3], Sec. 3.7)

In Problems 556 and 557 transform the given line integrals about closed (positive oriented) contours Γ into double integrals over the regions Ω bounded by these contours:

556. $\int_{\Gamma} ((1-x^2)y\,dx+x(1+y^2)\,dy)$.

557. $\int_{\Gamma} ((e^{xy}+2x\cos y)\,dx+(e^{xy}-x^2\sin y)\,dy)$.

558. Evaluate $\int_{\Gamma} ((xy+x+y)\,dx+(xy+x-y)\,dy)$, where Γ is the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

559. Compute the difference between the integrals

$$I_1=\int_{\Gamma_1}((x+y)^2\,dx-(x-y)^2\,dy),$$

$$I_2=\int_{\Gamma_2}((x+y)^2\,dx-(x-y)^2\,dy),$$

where Γ_1 is an arc of the parabola $y=x^2$, and Γ_2 is the line segment joining the points (0, 0), (1, 1).

560. Evaluate the integral

$$\int_{\Gamma}(-x^2y\,dx+xy^2\,dy),$$

where Γ is a positive oriented circle $x^2+y^2=R^2$.

561. Find the area of the figure Ω bounded by the astroid (Fig. 28): $x=a\cos^3 t$, $y=a\sin^3 t$.

562. Show that the work done by the force $\mathbf{a}=\{2xy,\ x^2\}$ in displacing a point of mass m depends only on its initial and final positions and is independent of the shape of the path. Compute the work A done by the force when displacing the given point from the position (1, 1) to (2, 5).

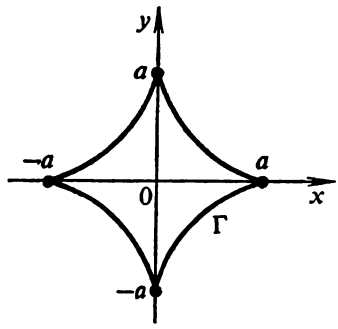

Fig. 28.

Sec. 8.6. Surface Integral of the First Kind (See [3], Secs. 2.11 and 3.8)

563. If a surface is represented parametrically

$$x=x(u,v),\quad y=y(u,v),\quad z=z(u,v)$$

or in vector form

$$\mathbf{r}(u,v)=x(u,v)\mathbf{i}+y(u,v)\mathbf{j}+z(u,v)\mathbf{k},$$

where $(u, v) \in \Omega$ and the functions x, y, z have continuous partial derivatives on the closure of Ω, then the area S of this surface is expressed by the double integral

$$S = \iint\limits_{\Omega} \sqrt{EG - F^2}\, du\, dv = \iint\limits_{\Omega} \sqrt{|\mathbf{r}_u|^2\, |\mathbf{r}_v|^2 - (\mathbf{r}_u, \mathbf{r}_v)^2}\, du\, dv,$$

where

$$E = |\mathbf{r}_u|^2 = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2,$$

$$G = |\mathbf{r}_v|^2 = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2,$$

$$F = (\mathbf{r}_u, \mathbf{r}_v) = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}.$$

Indeed, as we know (see [3], Sec. 2.11),

$$S = \iint\limits_{\Omega} |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv,$$

but

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{vmatrix},$$

therefore

$$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{\left[\frac{D(y, z)}{D(u, v)}\right]^2 + \left[\frac{D(z, x)}{D(u, v)}\right]^2 + \left[\frac{D(x, y)}{D(u, v)}\right]^2}.$$

Expanding the Jacobians under the radical sign, squaring them and performing all necessary algebraic transformations, we obtain the sought-for formula.

This formula for computing an area is convenient in a number of cases, particularly, when the vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ are orthogonal $((\mathbf{r}_u, \mathbf{r}_v) = 0)$.

In this case

$$S = \iint\limits_{\Omega} |\mathbf{r}_u| \cdot |\mathbf{r}_v|\, du\, dv.$$

Find the area of the surface:

$$x = u \cos v, \quad y = u \sin v, \quad z = 4v,$$

$$0 \leqslant u \leqslant 3, \quad 0 \leqslant v \leqslant \pi.$$

In the present case

$$\mathbf{r}_u = (\cos v, \sin v, 0), \quad \mathbf{r}_v = (-u \sin v, u \cos v, 4);$$
$$(\mathbf{r}_u, \mathbf{r}_v) = 0, \quad |\mathbf{r}_u|^2 = 1, \quad |\mathbf{r}_v|^2 = 16+u^2.$$

Therefore

$$S = \iint_{\Omega} 1 \cdot \sqrt{16+u^2}\, du\, dv = \int_0^{\pi} dv \int_0^3 \sqrt{16+u^2}\, du$$

$$= \pi \int_0^3 \sqrt{16+u^2}\, du = \pi\left[\frac{u}{2}\sqrt{16+u^2}+8 \ln\left(u+\sqrt{16+u^2}\right)\right]_0^3$$

$$= \frac{\pi}{2}(15+16 \ln^2).$$

564. Evaluate $\int_S (x^2+y^2)\, ds$, where S is a sphere $x^2+y^2+z^2 = a^2$.

565. Find the mass of the portion of the sphere $x^2+y^2+z^2 = a^2$ found in the first octant ($x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$) if the density of mass distribution over the sphere is equal to $\sqrt{x^2+y^2}$.

566. Compute $\int_S z\, dS$, where S is part of the surface of the helicoid

$$x = u \cos v, \quad y = u \sin v, \quad z = v \quad (0 \leqslant u \leqslant a, 0 \leqslant v \leqslant 2\pi).$$

Sec. 8.7. A Vector Flux Through an Oriented Surface (Surface Integral of the Second Kind) (See [3], Sec. 3.12)

The flux of the vector $\mathbf{a} = (P, Q, R)$ through an oriented surface S^*

$$\int_{S^*} (\mathbf{a}, d\mathbf{S}^*) = \int_S (\mathbf{a}, \mathbf{n})\, dS$$

$$= \int_S (P \cos(\mathbf{n}, x)+Q \cos(\mathbf{n}, y) + R \cos(\mathbf{n}, z))\, dS$$

is equal to the surface integral of the first kind of the scalar product $(\mathbf{a}, \mathbf{n})$, where $\mathbf{n}$ is the unit normal defining the orientation of S^*.

If a surface is given by the equation

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i}+y(u, v)\mathbf{j}+z(u, v)\mathbf{k} \quad ((u, v) \in \Omega),$$

then

$$dS = |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv, \quad \mathbf{n} = \pm \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|},$$

therefore

$$\int_{S^*} (\mathbf{a}, d\mathbf{S}^*) = \pm \int_{\Omega} ((\mathbf{a}, \mathbf{r}_u \times \mathbf{r}_v)\, du\, dv.$$

Hence it is seen that to the different sides of the surface S there respond the surface integrals of the second kind of the vector $\mathbf{a}$ differing in sign.

If the surface S is defined by an implicit equation $F(x, y, z) = 0$, then the direction cosines of the normal are determined by the formulas

$$\cos(\mathbf{n}, x) = \frac{\partial F}{\partial x}\Big/ D, \quad \cos(\mathbf{n}, y) = \frac{\partial F}{\partial y}\Big/ D, \quad \cos(\mathbf{n}, z) = \frac{\partial F}{\partial z}\Big/ D,$$

where

$$D = \pm |\operatorname{grad} F| = \pm \sqrt{F_x^2 + F_y^2 + F_z^2}.$$

The sign before the radical must be made consistent with the appropriate side of the surface S.

Surface integral of the second kind is also designated by the symbol

$$\int_{S^*} (\mathbf{a}, d\mathbf{S}^*) = \int_{S^*} (P\, dy\, dz + Q\, dx\, dz + R\, dx\, dy).$$

This notation is suitable for the case of explicit representation of a surface. If the surface S is simultaneously defined by the equations

$$x = f_1(y, z), \quad (y, z) \in \Omega_1;$$
$$y = f_2(x, z), \quad (x, z) \in \Omega_2;$$
$$z = f_3(x, y), \quad (x, y) \in \Omega_3,$$

then

$$\int_{S^*} (P\, dy\, dz + Q\, dx\, dz + R\, dx\, dy)$$
$$= \pm \iint_{\Omega_1} P(f_1(y, z), y, z)\, dy\, dz \pm \iint_{\Omega_2} Q(x, f_2(x, z), z)\, dx\, dz$$
$$\pm \iint_{\Omega_3} R(x, y, f_3(x, y))\, dx\, dy,$$

where sign (plus or minus) is taken depending on orientation of the surface (for instance, the first integral is taken with the plus or minus sign depending on what angle (acute or obtuse) is formed by the normal to S with the x-axis).

567. Compute $\int\limits_{S^*} z\,dx\,dy$, where S^* is the exterior side of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Solution. **Method 1.** The surface S is given by the explicit equation

$$z=\pm c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}.$$

The cosine of the acute angle formed by the outward normal with the z-axis for the upper half of the ellipsoid is determined by the formula

$$\cos(\mathbf{n},z)=1/\sqrt{1+z_x^2+z_y^2}$$

(and the lower half must be taken with the minus sign). Therefore we take the plus sign in the corresponding formula for the upper half S_1^* of the ellipsoid:

$$\int\limits_{S_1^*} z\,dx\,dy=\iint\limits_{\Omega} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dx\,dy.$$

Analogically, for the lower half S_2^*:

$$\int\limits_{S_2^*} z\,dx\,dy=-\iint\limits_{\Omega}-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dx\,dy$$

$$=c\iint\limits_{\Omega}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dx\,dy.$$

Thus,

$$\int\limits_{S^*} z\,dx\,dy=2c\iint\limits_{\Omega}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dx\,dy,$$

where $\Omega=\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant 1\right\}$. Computing the last integral, we obtain

$$\int\limits_{S^*} z\,dx\,dy=\frac{4}{3}\pi abc.$$

Method 2. Let us pass to parametric representation of the ellipsoid:

$$\begin{cases} x = a\cos u\cos v, \\ y = b\cos u\sin v, \\ z = c\sin u, \end{cases} \quad \Delta = \begin{cases} 0 \leqslant v \leqslant 2\pi \\ -\dfrac{\pi}{2} \leqslant u \leqslant \dfrac{\pi}{2} \end{cases}$$

Here

$$\mathbf{r}_u = \{-a\sin u\cos v,\ -b\sin u\sin v,\ c\cos u\},$$
$$\mathbf{r}_v = \{-a\cos u\sin v,\ b\cos u\cos v,\ 0\}.$$

In the right-handed system of coordinates the outward normal to the ellipsoid is determined by the equality

$$\mathbf{n} = -\frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}.$$

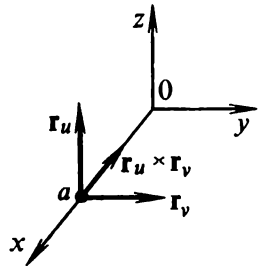

Fig. 29.

For example, let us take the point $x = a$, $y = 0$, $z = 0$ on the ellipsoid. This point corresponds to the parameters $u = v = 0$. At this point $\mathbf{r}_u = \{0, 0, c\}$, $\mathbf{r}_v = \{0, b, 0\}$ (Fig. 29). The vector product $\mathbf{r}_u \times \mathbf{r}_v$ together with the vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ must form a right-handed triple (that is, be oriented in the same way as the coordinate system), i.e. the vector $\mathbf{r}_u \times \mathbf{r}_v$ is in the direction of the negative x-axis (see Fig. 29). Hence,

$$\int\limits_{S^*} z\,dx\,dy = -\iint\limits_{\Delta} (\mathbf{a}, \mathbf{r}_u \times \mathbf{r}_v)\,du\,dv$$

$$= -\iint\limits_{\Delta} z\frac{D(x, y)}{D(u, v)}\,du\,dv = -\iint\limits_{\Delta} z\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} du\,dv$$

$$= abc\iint\limits_{\Delta} \sin^2 u\cos u\,du\,dv = 2\pi abc\int\limits_{-\pi/2}^{\pi/2} \sin^2 u\cos u\,du$$

$$= 4\pi abc\int\limits_{0}^{\pi/2} \sin^2 u d\sin u = \frac{4}{3}\pi abc.$$

568. Compute

$$\int\limits_{S^*} (x^2\,dy\,dz + y^2\,dx\,dz + z^2\,dx\,dy),$$

where S^* is the outer side of the hemisphere $x^2 + y^2 + z^2 = a^2$ $(z \geqslant 0)$.

569. Compute

$$\int_{S^*} (yz\,dy\,dz + xz\,dx\,dz + xy\,dx\,dy),$$

where S^* is the outer side of the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x+y+z = a$ (Fig. 30).

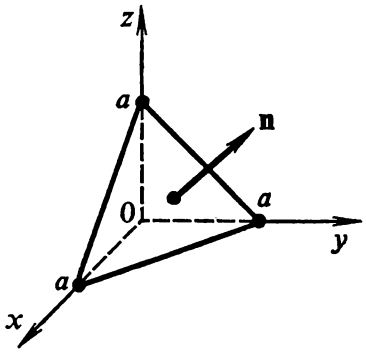

Fig. 30.

Sec. 8.8. Gauss-Ostrogradski Formula (See [3], Sec. 3.13)

570. Find the divergence of the vector $\mathbf{a} = (x^3, y^3, z^3)$.

571. Find the divergence of the vector $\mathbf{a} = f(r)\frac{\mathbf{r}}{r}$, where $r = |\mathbf{r}|$, $\mathbf{r} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, f is a differentiable function.

572. Let $u = x^2+y^2+z^2$. Find div (grad u).

573. Compute curl $\mathbf{a}$ if: (a) $\mathbf{a} = \mathbf{r}$, (b) $\mathbf{a} = f(r)\,\mathbf{c}$, where $\mathbf{c} = c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}$ is a constant vector, $r = |\mathbf{r}| = |x\mathbf{i}+y\mathbf{j}+z\mathbf{k}|$.

574. Using the Ostrogradski-Gauss formula

$$\iiint_G \operatorname{div} \mathbf{a}\, dG = \int_S (\mathbf{a}, \mathbf{n})\, dS,$$

where $\mathbf{n}$ is the unit vector of the outward normal to the surface S which is the boundary of the solid G, transform the following surface integrals of the second kind:

(a) $\int_S (xy\,dx\,dy + yz\,dy\,dz + xz\,dx\,dz)$;

(b) $\int_S (x^2\,dy\,dz + y^2\,dx\,dz + z^2\,dx\,dy)$;

(c) $\int_S \frac{x\cos\alpha + y\cos\beta + z\cos\gamma}{\sqrt{x^2+y^2+z^2}}\,dS$;

(d) $\int_S \left(\frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\cos\beta + \frac{\partial u}{\partial z}\cos\gamma\right) dS$.

In Problems 575 to 577 evaluate the indicated surface integrals with the aid of the Ostrogradski-Gauss formula.

575.

$$\int_S (x\,dy\,dz + y\,dx\,dz + z\,dx\,dy),$$

where S is the outer side of the pyramid bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x+y+z = 1$.

576. $\int_S z^2\,dx\,dy$, where S is the outer side of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

577. $\int_S z^4\,dx\,dy$, where S is the outer side of the ellipsoid from Problem 576.

Sec. 8.9. Stokes' Formula (See [3], Sec. 3.15)

Stokes' formula, establishing relation between the circulation of the vector $\mathbf{a} = (P, Q, R)$ around a contour Γ and the flux of the vector curl $\mathbf{a}$ through an oriented surface S^* (with the rim Γ), can be written in the following expanded form:

$$\int_\Gamma (\mathbf{a}\,d\mathbf{e}) = \int_\Gamma (P\,dx + Q\,dy + R\,dz)$$

$$= \int_S \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} dS = \int_S (\mathbf{n}, \operatorname{curl}\mathbf{a})\,dS,$$

where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the normal $\mathbf{n}$ to the surface S. The contour Γ is oriented in accordance with the orientation of S^* (Fig. 31).

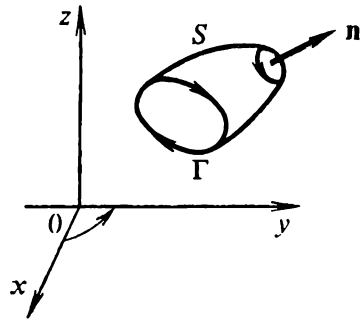

Fig. 31.

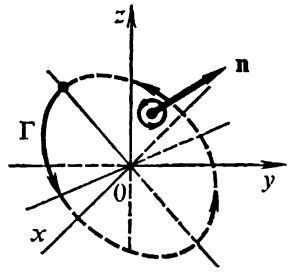

Fig. 32.

578. Applying Stokes' formula, compute the line integral (circulation)

$$\int_\Gamma (y\,dx+z\,dy+x\,dz),$$

where Γ is a circle $x^2+y^2+z^2=b^2$, $x+y+z=0$ traced anticlockwise when viewed from the side of the positive x-axis.

Solution. This tracing of the contour Γ corresponds to the orientation of the piece of the plane $x+y+z=0$ lying inside the sphere $x^2+y^2+z^2=b^2$ by the normal directed upward to the right (as is shown in Fig. 32):

$$\cos\alpha=\cos\beta=\cos\gamma=1/\sqrt{3}$$

$(F(x, y, z)\equiv x+y+z=0,\ F'_x=F'_y=F'_z=1,\ |\text{grad}\,F|=\sqrt{3})$.

In this case for the vector $\mathbf{a}$: $P=y$, $Q=z$, $R=x$, therefore

$$\int_\Gamma (y\,dx+z\,dy+x\,dz)=-\int_S (\cos\alpha+\cos\beta+\cos\gamma)\,dS=-\frac{3}{\sqrt{3}}\int_S dS,$$

where S is a circle of radius b lying in the plane $x+y+z=0$. The surface integral of the first kind of a unit function is obviously equal to the area of she surface, therefore

$$\int_\Gamma (y\,dx+z\,dy+x\,dz)=-\frac{3}{\sqrt{3}}\pi b^2=-\sqrt{3}\pi b^2.$$

579. Compute, by Stokes' formula and directly, the circulation

$$\int_\Gamma ((y-z)\,dx+(z-x)\,dy+(x-y)\,dz),$$

where Γ is an ellipse: $x^2+y^2=1$, $x+z=1$ (Fig. 33) traced anticlockwise when viewed from the direction of the positive z-axis.

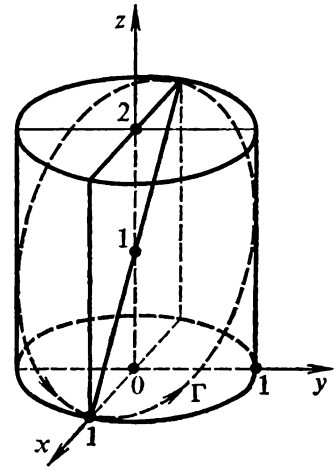

Fig. 33.

580. Using Stokes' formula, compute the integral

$$\int_\Gamma ((y+z)\,dx+(x+z)\,dy+(x+y)\,dz),$$

where Γ is a circle: $x^2+y^2+z^2=b^2$, $x+y+z=0$ (see Problem 578).

Chapter 9

FOURIER SERIES AND FOURIER INTEGRAL (See [3], Chapter 4)

Sec. 9.1. Trigonometric Series (See [3], Secs. 4.1 and 4.2)

581. Construct the graphs of the partial sums $S_1(x)$, $S_2(x)$ of the series

$$\sum_{k=1}^{\infty} \frac{\cos kx}{k},$$

582. Show that the function

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{k^3}$$

is continuous and has a continuous derivative on $(-\infty, \infty)$.

583. Find out at what points of the period the series

$$\sum_{k=1} \frac{\cos kx}{k^{\alpha}} \quad (\alpha > 0)$$

is convergent.

584. How many times is it possible to differentiate the series

$$\sum_{k=1}^{\infty} \frac{\sin kx}{2^k}?$$

585. Find the domain of convergence of the series

$$\sum_{k=1}^{\infty} \frac{\sin kx}{k!}.$$

Sec. 9.2. Fourier Series (See [3], Secs. 4.3, 4.4, and 4.6)

586. Expand the given function $f(x)$ into a Fourier series with period 2π if

(a) $f(x) = x$ on $(-\pi, \pi)$;

(b) $f(x) = \frac{\pi - x}{2}$ on $(0, 2\pi)$;

(c) $f(x) = |x|$ on $(-\pi, \pi)$;

(d) $f(x) = \begin{cases} 0, & -\pi < x \leqslant 0, \\ x, & 0 < x < \pi; \end{cases}$

(e) $f(x) = \begin{cases} x, & -\pi < x \leqslant 0, \\ 0, & 0 < x < \pi; \end{cases}$

(f) $f(x) = \sin ax, \quad -\pi < x < \pi;$

(g) $f(x) = \cosh ax, \quad -\pi < x < \pi.$

587. Investigate the convergence of Fourier's series for the periodic function

$$f(x) = \begin{cases} 0, & -\pi < x \leqslant 0, \\ x, & 0 < x < \pi \end{cases}$$ (see Problem 586, (d)).

Solution. The given function is bounded, piecewise continuous and piecewise monotone on $[0, 2\pi]$. At the point of discontinuity $x = \pi$ it is not defined (see Fig. 34), i.e. it does not satisfy Dirichlet's condition. But the value of Fourier's coefficients is independent of what values are attained by the function $f(x)$ at a particular point. Therefore, let us define the function $f(x)$ at the point $x = \pi$, putting

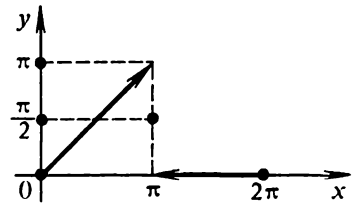

Fig. 34.

$$f(\pi) = \frac{f(\pi+0)+f(\pi-0)}{2} = \frac{0+\pi}{2} = \frac{\pi}{2}.$$

Then the Fourier series of this function (see Problem 586, (d))

$$\frac{\pi}{4} - \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{\cos(2k+1)x}{(2k+1)^2} + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sin kx}{k}$$

converges at all the points $x \in [0, 2\pi]$ to the extended function, in particular, it converges to $\pi/2$ at the point $x = \pi$, i.e.

$$\frac{\pi}{2} = \frac{\pi}{4} + \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2},$$

whence

$$\frac{\pi^2}{8} = \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}.$$

588. Expand the indicated functions into Fourier's series in terms of sines and cosines:

(a) $f(x) = x, \quad 0 < x < \pi;$

(b) $f(x) = \frac{\pi-2x}{4}, \quad 0 < x < \pi.$

589. Expand into Fourier's series the function $f(x)$ with period $2l$ defined by the formula $f(x) = |x|$ on $(-l, l)$.

Sec. 9.3. Orthogonal Systems of Functions (See [3], Secs. 4.5, 4.8 and 4.9)

590. Find the scalar product of the functions:

(a) $f(x) = x$, $\varphi(x) = \sin x$, $0 \leqslant x \leqslant \pi$;

(b) $f(x) = \sin x$, $\varphi(x) = \sin x$, $0 \leqslant x \leqslant \pi/2$.

591. Find the norm

$$\|f\| = \left(\int_a^b |f(x)|^2 \, dx \right)^{1/2}$$

of the indicated functions $f(x)$ $(a \leqslant x \leqslant b)$:

(a) $f(x) = x^2$, $0 \leqslant x \leqslant 1$;

(b) $f(x) = \cos x$, $0 \leqslant x \leqslant \pi$;

(c) $f(x) = e^x$, $0 \leqslant x \leqslant 1$;

(d) $f(x) = 1 - x$, $0 \leqslant x \leqslant 1$.

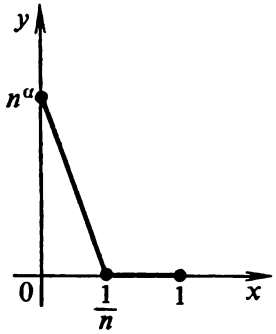

Fig. 35.

592. Let there be given a sequence of functions (Fig. 35)

$$f_n(x) = \begin{cases} n^\alpha - n^{1+\alpha}x, & 0 \leqslant x \leqslant 1/n, \\ 0, & 1/n < x \leqslant 1. \end{cases}$$

For what $\alpha \geqslant 0$ is this sequence convergent to zero in the mean?

593. Investigate the sequence

$$f_n(x) = \begin{cases} \sin \pi n^\alpha x, & 0 \leqslant x \leqslant n^{-\alpha}, \\ 0, & n^{-\alpha} < x < \pi \end{cases}$$

for uniform and root-mean-square convergence.

594. Prove that the set of Legendre polynomials

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \quad (n = 0, 1, \ldots)$$

is a complete set of orthogonal functions on the interval $(-1, 1)$, i.e.

$$\int_{-1}^{1} P_m(x) P_n(x) \, dx = 0 \quad (m \neq n),$$

and satisfy the condition

$$\int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}.$$

Solution. Note that from the definition of the Legendre polynomial $P_n(x)$ it follows that its coefficient of x^n is equal to $\frac{(n+1)\ldots(2n-1)2n}{2^n n!}$. Hence

$$\frac{d^n P_n(x)}{dx^n} = \frac{(n+1)(n+2)\ldots 2n}{2^n}.$$

Further, it is obvious that if $k < n$, then

$$\frac{d^k}{dx^k}(x^2-1)n = (x^2-1)^{n-k} A(x),$$

where $A(x)$ is a certain polynomial. Therefore for $k < n$

$$\frac{d^k}{dx^k}(x^2-1)^n\Big|_{x=\pm 1} = 0.$$

Let, for the sake of definiteness, $m > n$. Integrating by parts n times, we have

$$\int_{-1}^{1} P_n(x) P_m(x)\,dx = c\int_{-1}^{1} P_n(x)\frac{d^m}{dx^m}(x^2-1)^m\,dx$$

$$= cP_n(x)\frac{d^{m-1}}{dx^{m-1}}(x^2-1)^m\Big|_{-1}^{1} - c\int_{-1}^{1} P_n'(x)\frac{d^{m-1}}{dx^{m-1}}(x^2-1)^m\,dx$$

$$= (-1)c\int_{-1}^{1} P_n'(x)\frac{d^{m-1}}{dx^{m-1}}(x^2-1)^m\,dx =$$

$$\ldots = (-1)^n c\int_{-1}^{1} P_n^{(n)}(x)\frac{d^{m-n}}{dx^{m-n}}(x^2-1)^m\,dx$$

$$= (-1)^n c\frac{(n+1)\ldots 2n}{2^n}\int_{-1}^{1}\frac{d^{m-n}}{dx^{m-n}}(x^2-1)^m\,dx$$

$$= c_1\frac{d^{m-n-1}}{dx^{m-n-1}}(x^2-1)^m\Big|_{-1}^{1} = 0,$$

where

In a similar way, we obtain

$$\int_{-1}^{1} P_n(x)\,P_n(x)\,dx = \frac{(-1)^n\,(n+1)\,\ldots\,2n}{n!\,2^{2n}} \int_{-1}^{1} (x^2-1)^n\,dx$$

$$= 2\,\frac{(n+1)\,\ldots\,2n}{n!\,2^{2n}} \int_{0} (1-x^2)^n\,dx.$$

Let us evaluate the integral

$$I_n = \int_{0}^{1} (1-x^2)^n\,dx, \qquad (n = 0, 1, \ldots).$$

Obviously, $I_0 = 1$, $I_1 = 2/3$. Integrating by parts, we obtain the following recurrence formula:

$$I_n = \frac{2n}{2n+1}\,I_{n-1}.$$

Hence,

$$I_n = \frac{2\cdot4\cdot6\,\ldots\,2n}{3\cdot5\cdot7\,\ldots\,(2n+1)} = \frac{2^n\cdot n!}{3\cdot5\,\ldots\,(2n+1)} = \frac{2^{2n}(n!)^2}{(2n)!\,(2n+1)}.$$

Thus

$$\int_{-1}^{1} P_n^2(x)\,dx = 2\,\frac{(n+1)\,\ldots\,(2n-1)2n}{2^{2n}\,n!} \cdot \frac{2^{2n}(n!)^2}{(2n)!\,(2n+1)} = \frac{2}{2n+1}.$$

Sec. 9.4. Fourier Integral (See [3], Secs. 4.12 and 4.14)

595. Find the cosine transform of the function

$$f(x) = \begin{cases} 1, & 0 < x < a, \\ 0, & x > a. \end{cases}$$

596. Find the function defined on $(0, \infty)$ whose cosine transform is equal to

$$\sqrt{\frac{2}{\pi}}\,\frac{\sin as}{s}.$$

597. Find the cosine transform of the function

$$f(x) = \begin{cases} \cos x, & 0 < x < a, \\ 0, & x > a. \end{cases}$$

598. Find the sine transform of the function

$$f(x) = e^{-sx}/x.$$

599. Evaluate the following integrals:

(a) $\int_0^\infty \frac{\sin xs\,dx}{x(a^2+x^2)}$ $(a > 0,\ s \geqslant 0)$;

(b) $\int_0^\infty e^{-4x} \sin 3x \cos 2x\,dx$;

(c) $\int_0^\infty e^{-3x} \cos 3x \cos 4x\,dx$.

Chapter 10

EQUATIONS OF MATHEMATICAL PHYSICS

(See [3], Chapter 5)

600. Let a function $f(\theta)$ with period 2π be defined on $(-\pi, \pi)$ by the equality $f(\theta) = |\theta|$. Construct the harmonic in the unit circle function generated by these boundary values, and find out at what rate, in the mean square, the function $u(\varrho, \theta)$ tends to its boundary values $f(\theta)$ for $\varrho \to 1-0$.

601. Find the solution of the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad (0 \leqslant x \leqslant \pi, \ t > 0)$$

for the initial condition

$$u(x, 0) = f(x) = \frac{\pi - 2x}{4}, \quad 0 \leqslant x \leqslant \pi,$$

and the boundary condition

$$u(0, t) = u(\pi, t) = 0.$$

Estimate the integral

$$A = \left(\int_0^{\pi} |u(x, t) - f(x)|^2 \, dx \right)^{1/2}$$

for $t \to +0$, i.e. find out the character of convergence in the mean of the solution $u(x, t)$ to $f(x)$ as $t \to +0$.

602. Find the solution of the string-oscillation equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

for the initial conditions

$$u(x, 0) = f(x), \quad \frac{\partial u(x, 0)}{\partial t} = f(x)$$

$$\left(f(x) = \begin{cases} x, & 0 \leqslant x \leqslant \pi/2, \\ \pi - x, & \pi/2 < x \leqslant \pi, \end{cases} \right)$$

and for the boundary conditions

$$u(0, t) = u(\pi, t) = 0.$$

603. Solve the equation of oscillation of an endless string

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

for the initial conditions

$$u(x, 0) = u_t'(x, 0) = \frac{1}{1+x^2} \quad (-\infty < x < \infty).$$

604. Let $u(x, y)$ be a harmonic in the upper half-plane function attaining the values of $f(x)$ for $y = 0$. Prove that if $f(x)$ $\forall x$ satisfies the condition

$$|f(x+t)-f(x)| \leqslant L\,|t|^{\alpha}, \quad 0 < \alpha < 1,$$

then

$$|u(x, y)-f(x)| \leqslant cy^{\alpha} \qquad (y > 0),$$

where c is a constant independent of x and y.

605. Find the stationary distribution of temperature $u(x, y)$ in the upper half-plane and the isotherm (level line) $u = 1/2$ if on the x-axis the temperature

$$f(x) = \begin{cases} 1, & |x| \leqslant l, \\ 0, & |x| > l \end{cases}$$

is maintained.

Hint. The function $u(x, y)$ is harmonic in the upper half-plane.

606. Find the temperature distribution in an endless rod if the initial temperature distribution was

$$f(x) = \exp(-x^2)$$

(put $a = 1$ in the heat equation).

607. Prove that the Legendre polynomials $y = P_n(x)$ (see Problem 594) satisfy the differential equation

$$\frac{d}{dx}(1-x^2)\frac{dy}{dx}+n(n+1)y = 0.$$

Chapter 11

FUNCTIONS OF A COMPLEX VARIABLE

(See [3], Chapter 6; [1], Chapter 5)

Sec. 11.1. General Concepts (See [1], Sec. 5.3)

608. Find the moduli of the following complex numbers:

(a) $z = 4+3i$; (b) $z = \cos\alpha - i\sin\alpha$; (c) $z = -2+2\sqrt{3}i$.

609. Write in the trigonometric form the following complex numbers: (a) $z = -1-i\sqrt{3}$; (b) $z = -\sqrt{2}+i\sqrt{2}$.

610. Represent in the exponential form the following complex numbers:

(a) $z = -2$; (b) $z = i$; (c) $z = -1-i\sqrt{3}$;

(d) $z = \sin\alpha - i\cos\alpha$ $(\pi/2 < \alpha < \pi)$.

611. Compute $(-1+i\sqrt{3})^{60}$.

Solution. Represent the number $z = -1+i\sqrt{3}$ in the exponential form

$$-1+i\sqrt{3} = 2\exp\left(i\frac{2\pi}{3}\right).$$

Hence,

$$(-1+i\sqrt{3})^{60} = 2^{60}\exp(40\pi i) = 2^{60}.$$

612. Compute: (a) $(\sqrt{3}-3i)^6$; (b) $\left(\frac{1-i}{1+i}\right)^4$.

613. Find all the values of the radicals:

(a) $\sqrt[4]{1-i}$; (b) $\sqrt[4]{1}$.

614. Let $\operatorname{Re} w = x$, $\operatorname{Im} w = y^2$ $(w \neq 0)$. Find $\bar{w}$, $1/w$.

615. Prove the indicated equalities:

(a) $z+\bar{z} = 2\operatorname{Re} z$; (b) $z-\bar{z} = 2i\operatorname{Im} z$; (c) $|\bar{z}| = |z|$.

616. What curves are given by the following equations:

(a) $|z-z_0| = r$, $r > 0$; (b) $|z+c|+|z-c| = 2a$, $a > c$—real numbers; (c) $\operatorname{Re}(1/w) = 1/2$, $\operatorname{Im}(1/w) = 1/4$, $w = x+2yi$; (d) $\operatorname{Im} z^2 = 2$; (e) $\operatorname{Im}\left(\frac{z}{x^2}\right) = 1$?

617. Find the images of points z_0 for the indicated mappings:

(a) $w = z^2$, $z_0 = i$; (b) $= w = \bar{z}/z$, $z_0 = 2+3i$.

618. Into what curve is the circle $|z| = \sqrt{2}$ mapped with the aid of the function $w = z^2$?

Solution. We have $\operatorname{Re} w = x^2-y^2$, $\operatorname{Im} w = 2xy$. Eliminating x and y form the system

$$\left.\begin{aligned} u &= x^2-y^2, \\ v &= 2xy, \\ x^2+y^2 &= 2, \end{aligned}\right\}$$

we obtain

$$u^2+v^2 = (x^2+y^2)^2 = 4,$$

i.e. this is a circle of radius 2 with centre at the origin in the uv-plane (in the w-plane). Note that the circle $|w| = 2$ is described twice when the point z traverses the complete circle $|z| = \sqrt{2}$, since

$$\operatorname{Arg} w = 2 \operatorname{Arg} z+2k\pi.$$

The present problem can also be solved by applying another technique. The equation of the circle $|z| = \sqrt{2}$ may be written in the form

$$z = \sqrt{2}e^{i\varphi},$$

where $0 \leqslant \varphi < 2\pi$. Therefore

$$w = z^2 = (\sqrt{2}e^{i\varphi})^2 = 2e^{i2\varphi}.$$

Hence it follows that the circle $|z| = \sqrt{2}$ goes into the circle $|w| = 2$ in the mapping $w = z^2$, the circle $|w| = 2$ being described twice when the point z traverses the circle $|z| = \sqrt{2}$ once in the positive direction (anticlockwise).

619. Find out on what lines of the w-plane the following curves in the z-plane are mapped with the aid of the function $w = 1/z$:

(a) $|z| = 1/2$; (b) $\arg z = \pi/4$, (c) $\operatorname{Re} z = 0$.

Sec. 11.2. Limit of a Function. Derivatives (See [3], Secs. 6.1 and 6.2)

620. Find the limit of the function $w = \bar{z}$ at the point $z = i$.

621. Find out whether the limit of the function $w = \bar{z}/z$ exists at the point $O = (0, 0)$.

622. Will the function $w = \operatorname{Re} z$ be continuous in the z-plane?

623. Which of the following functions have a derivative:

(a) $w = z^2$, (b) $w = \operatorname{Re} z$, (c) $w = \bar{z}$; (d) $w = z \cdot \bar{z}$?

624. Find the modulus of expansion and the angle of rotation at the point $z_0 = 0$ in the mapping $w = \sin z$.

625. At what points is the mapping $w = f(z)$ conformal:

(a) $w = z^3$; (b) $w = \cos z$; (c) $w = ze^z$?

Sec. 11.3. Cauchy-Riemann Conditions. Harmonic Functions (See [3], Secs. 6.3 and 6.4)

626. Find out which of the given functions are analytic:

(a) $w = ze^z$; (b) $w = \bar{z}z^2$; (c) $w = \sin 3z$; (d) $w = \cosh 2z$.

627. Restore the analytic function $f(z) = u + iv$ by its known real part $u(x, y)$:

(a) $u = x^2 - y^2 + 2x$; (b) $u = x^2 - y^2 + xy$;

(c) $u = r\varphi \cos \varphi + r \ln r \sin \varphi$ $(z = re^{i\varphi})$.

628. Find all the harmonic functions of the form $u = \varphi(x^2 + y^2)$.

Solution. We have

$$u''_{x^2} = 4x^2\varphi''(x^2+y^2) + 2\varphi'(x^2+y^2),$$
$$u''_{y^2} = 4y^2\varphi''(x^2+y^2) + 2\varphi'(x^2+y^2),$$

whence

$$\Delta u = 4(x^2+y^2)\,\varphi''(x^2+y^2) + 4\varphi'(x^2+y^2).$$

Thus, in order for the function u to be harmonic ($\Delta u = 0$), the following equality must be fulfilled:

$$(x^2+y^2)\,\varphi''(x^2+y^2) + \varphi'(x^2+y^2) = 0.$$

Setting $x^2 + y^2 = t$, we obtain

$$\frac{\varphi''(t)}{\varphi'(t)} = -\frac{1}{t}, \quad \frac{d \ln \varphi'(t)}{dt} = -\frac{1}{t},$$
$$\varphi'(t) = C/t, \quad \varphi(t) = C \ln t + C_1.$$

Hence, the harmonic functions have the form

$$u = C \ln (x^2+y^2) + C_1,$$

where C and C_1 are arbitrary constants.

629. Find all the harmonic functions of the form $u = \varphi(y/x)$.

Sec. 11.4. Simplest Conformal Mappings (See [3], Secs. 6.2 and 6.15)

630. Find the conformal mapping transforming the upper half-plane onto itself.

631. Map the unit circle $|z| \leqslant 1$ onto the upper half-plane so that the points $z_1 = -i$, $z_2 = 1$, $z_3 = i$ go into the respective points $w_1 = -1$, $w_2 = 0$, $w_3 = 1$.

Solution. The desired mapping is realized by the linear-fractional function

$$\frac{w+1}{w-0} : \frac{1+1}{1-0} = \frac{z+i}{z-1} : \frac{i+i}{i-1} \quad \text{or} \quad w = \frac{i(1-z)}{1+z}.$$

The inverse function

$$z = \frac{i-w}{w+i}$$

maps the upper half-plane onto the unit circle so that w_k go into z_k $(k = 1, 2, 3)$.

632. Map conformally the angle $0 < \varphi < \pi/4$ onto the upper half-plane (Fig. 36).

633. Map conformally the strip $0 \leqslant y < \pi$: (a) onto the upper half-plane; (b) onto the entire plane.

634. Map the vertical strip $0 \leqslant x \leqslant \pi/4$ onto the unit circle $|w| \leqslant 1$ (Fig. 37).

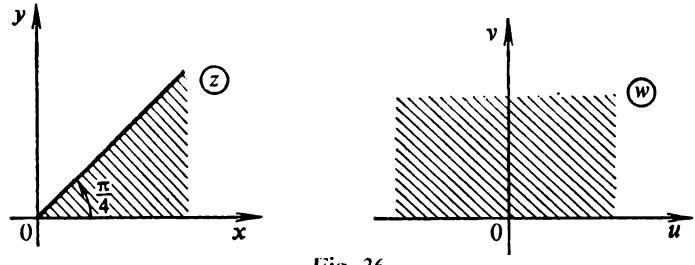

Fig. 36.

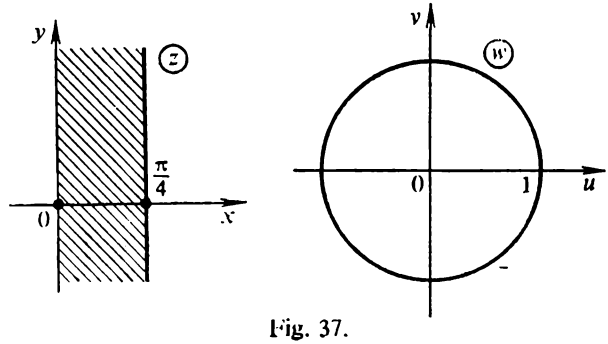

Fig. 37.

Solution. Let us turn the vertical strip into a horizontal one. To this end, let us perform a rotation through $\pi/2$ which is realized by the function

$$z^* = z \exp\left(i\frac{\pi}{2}\right) = iz.$$

In the plane $z^* = x^* + iy^*$ we obtain a horizontal strip $0 \leqslant y^* \leqslant \pi/4$. Let us expand this strip fourfold:

$$z' = 4z^* = 4iz.$$

In the plane z' we obtain a horizontal strip $0 \leqslant y' \leqslant \pi$. We then map this strip onto the upper half-plane with the aid of the exponential function

$$z'' = e^{z'} = e^{4iz}.$$

Finally, we map this half-plane onto the unit circle $|w| \leqslant 1$, for instance, with the aid of the function from Problem 631:

$$w = \frac{i-z''}{z''+i} = \frac{i-\exp(4iz)}{i+\exp(4iz)}.$$

635. Find the entire linear function which maps the triangle with vertices at the points 0, 1, i in the z-plane onto a triangle with respective vertices $1+i$, 0, 2 in the w-plane.

636. Find the conformal mapping of the circle $|z| < 5$ onto the circle $|w| < 1$ so that the points 5, $4+3i$, -5 go into the points 1, i, -1.

Sec. 11.5. Integration of Functions of a Complex Variable (See [3], Sec. 6.6)

637. Compute the integral

$$\int_C (1+i-2\bar{z})\,dz$$

along the lines connecting the points $z_1 = 0$, $z_2 = 1+i$: (a) along a straight line; (b) along the parabola $y = x^2$; (c) along a polygonal line $z_1z_3z_2$ where $z_3 = 1$ (Fig. 38).

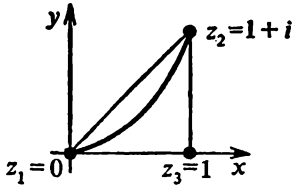

Fig. 38.

638. Evaluate $\int_0^{\pi i} z \cosh z\,dz$.

639. Evaluate $\int_0^{\pi(1+i)} z \cosh z\,dz$.

If a function $f(z)$ is analytic in a simply connected domain D, then the function

$$F(z) = \int_{z_0}^{z} f(\xi)\, d\xi$$

is also analytic in D, and

$$F'(z) = f(z).$$

Indeed,

$$F'(z) = \lim_{h \to 0} \frac{F(z+h)-F(z)}{h} = \lim_{h \to 0} \frac{1}{h} \int_{z}^{z+h} f(\xi)\, d\xi$$

$$= \lim_{h \to 0} \frac{1}{h} \int_{z}^{z+h} [f(z)+\eta(\xi)]\, d\xi = f(z) + \lim_{h \to 0} \frac{1}{h} \int_{z}^{z+h} \eta(\xi)\, d\xi = f(z),$$

where $\eta(\xi) \to 0$ as $\xi \to z$, since f is continuous at the point z. For small h, $|\eta(\xi)| < \varepsilon$, therefore for such h

$$\left| \frac{1}{h} \int_{z}^{z+h} \eta(\xi)\, d\xi \right| \leqslant \frac{\varepsilon}{|h|} \left| \int_{z}^{z+h} d\xi \right| = \frac{\varepsilon}{|h|} \cdot |h| = \varepsilon.$$

It is understood here that integration is performed along the straight line connecting the points z and $z+h$. We are allowed to proceed so, since $f(z)$ is analytic and, consequently, the integral is independent of the path of integration.

Thus, we have proved that $F'(z) = f(z)$. The function $F(z)$ is called the *antiderivative* (or *primitive*) for $f(z)$.

The same as in the case of a real variable, we can establish that two arbitrary antiderivatives for the function $f(z)$ differ by a constant term.

Hence it follows that if $\Phi(z)$ is an antiderivative for $f(z)$, then

$$\int_{z_0}^{z} f(\xi)\, d\xi = \Phi(z) - \Phi(z_0)$$

which is known as the Newton-Leibniz formula.

640. Using the Newton-Leibniz formula, compute the indicated integrals:

(a) $\int_{1-i}^{2+i} (3z^2+2z)\, dz$; (b) $\int_{0}^{i} z \sin z\, dz$.

Sec. 11.6. Cauchy's Integral Formula (See [3], Secs. 6.7 and 6.8)

We know that if $f(z)$ is analytic in a domain $\bar{D}$ bounded by a piecewise smooth contour C, then there takes place Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\xi)\, d\xi}{\xi - z} \quad (z \in D). \tag{1}$$

641. Evaluate the integral

$$I = \int_C \frac{\exp(z^2)\, dz}{z^2 - 6z},$$

if: (a) C: $|z-2| = 1$; (b) C: $|z| = 1$; (c) C: $|z-6| = 1$ (Fig. 39).

642. Compute the integrals:

(a) $\displaystyle\int_{|z-1|=3/2} \frac{\exp(z^2)\, dz}{z(z+1)}$; (b) $\displaystyle\int_{|z+1|=1/2} \frac{\exp(z)\, dz}{z(z+1)}$.

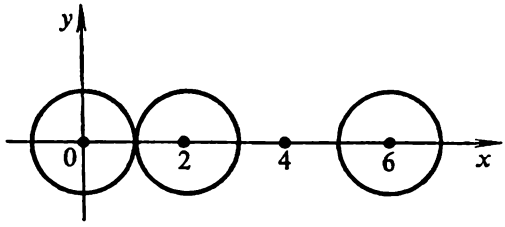

Fig. 39.

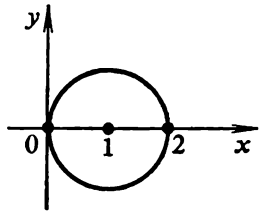

Fig. 40.

643. Evaluate the integral

$$I = \int_{|z-1|=1} \frac{\sin \pi z}{(z^2-1)^2}\, dz.$$

Solution. The function $f(z) = \dfrac{\sin \pi z}{(z+1)^2}$ is analytic on the circle $|z-1| \leqslant 1$ (Fig. 40). Therefore, using the formula

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)\, dz}{(z-z_0)^{n+1}} \tag{2}$$

for $n = 1$, where C is the circle $|z-1| = 1$, we obtain

$$I = \int_{|z-1|=1} \frac{\sin \pi z}{(z+1)^2} \frac{dz}{(z-1)^2} = 2\pi i \left(\frac{\sin \pi z}{(z+1)^2}\right)' \Big|_{z=1}$$

$$= 2\pi i \frac{\pi(z+1)\cos \pi z - 2 \sin \pi z}{(z+1)^3}\Big|_{z=1} = -2\pi i \frac{\pi}{4} = -\frac{\pi^2 i}{2}.$$

644. Evaluate the integral I round the circle $|z| = 2$

$$I = \int\limits_{|z|=2} \frac{\cosh z\, dz}{(z+1)^3 (z-1)}.$$

Hint. Construct the contours C_1 and C_2 incorporating, respectively, the points $z = -1$ and $z = 1$ and lying inside the circle $|z| = 2$. Then $\int\limits_{|z|=2} \ldots = \int\limits_{C_1} \ldots + \int\limits_{C_2} \ldots$, and then apply formulas (1) and (2).

Sec. 11.7. Series in a Complex Domain (See [3], Secs. 6.9 and 6.10)

In Problems 645 to 651 find the radius R of convergence of the indicated series:

645. $\sum\limits_{k=0}^{\infty} \frac{z^k}{k!}$. **646.** $\sum\limits_{k=0}^{\infty} k!\, z^k$. **647.** $\sum\limits_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{k!}$.

648. $\sum\limits_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}$. **649.** $\sum\limits_{k=0}^{\infty} (-1)^k z^k$. **650.** $\sum\limits_{n=0}^{\infty} e^{in} z^n$.

651. $\sum\limits_{n=0}^{\infty} (n+i) z^n$.

In Problems 652 and 653 expand the given function in a Taylor's series in the neighbourhood of $z = 0$:

652. $\frac{1}{(1+z)^2}$. **653.** $\frac{1}{(1+z)(z-2)}$.

654. Expand the function $f(z) = \frac{1}{5+2z}$ in powers of $(z-3)$.

Solution. Transform the given function in the following way:

$$\frac{1}{5+2z} = \frac{1}{5+2(z-3+3)} = \frac{1}{11+2(z-3)} = \frac{1}{11} \cdot \frac{1}{1+\frac{2}{11}(z-3)}.$$

Substituting in the expansion

$$\frac{1}{1+z} = 1 - z + z^2 - \ldots + (-1)^n z^n + \ldots$$

$\frac{2}{11}(z-3)$ for z, we get

$$\frac{1}{5+2z} = \frac{1}{11}\left\{1 - \frac{2}{11}(z-3) + \frac{2^2}{11^2}(z-3)^2 - \ldots\right\}$$

$$= \frac{1}{11} - \frac{2}{11^2}(z-3) + \frac{2^2}{11^3}(z-3)^2 - \ldots.$$

The last series converges for

$$\frac{2}{11}|z-3|<1, \quad \text{or} \quad |z-3|<\frac{11}{2}.$$

Hence, the radius of its convergence $R = 11/2$.

655. Expand the function $f(z) = 1/(3-2z)$ in powers of $(z-3)$.

656. Determine the domain of convergence of the given series:

(a) $\sum_{n=1}^{\infty} \frac{(\sqrt{2}+i\sqrt{2})^n}{z^n}$; (b) $\sum_{n=1}^{\infty} \frac{1}{4^n(z+1)^n}$.

657. Expand the function $f(z) = \frac{1}{(z-1)(z-2)}$ in Laurent's series in powers of z in the following domains: (a) $0<|z|<1$; (b) $1<|z|<2$; (c) $|z|>2$.

658. Expand the function

$$f(z) = \frac{1}{(z^2-1)^2}$$

into a Laurent's series in the annulus $0<|z-1|<2$.

In Problems 659 and 660 expand the given functions in a Laurent's series in the neighbourhood of $z = 0$:

659. $\frac{\sin z}{z^2}$. **660.** $z^3 \exp(1/z)$.

Sec. 11.8. Isolated Singularities. Residues (See [3], Secs. 6.11-6.13)

661. Determine the character of the singular point $z = 0$ for the indicated functions:

(a) $f(z) = (e^z-1)/z$; (b) $f(z) = 1/z^4$, (c) $f(z) = \exp(1/z^2)$.

662. Find all the singular points and determine their character in the given functions:

(a) $\frac{z}{\sin z}$; (b) $\cos\frac{1}{z}$; (c) $z\sin\frac{1}{z}$; (d) $\tanh z$.

663. Find the residue of the function $\frac{A}{z-a}$ at the point $z=\infty$.

Solution. We have

$$\frac{A}{z-a} = \frac{A}{z\left(1-\frac{a}{z}\right)} = \frac{A}{z}\sum_{k=0}^{\infty}\left(\frac{a}{z}\right)^k = \frac{A}{z}+\frac{Aa}{z^2}+\ldots,$$

consequently,

$$\operatorname*{Res}_{z=\infty}\frac{A}{z-a} = -A.$$

664. Find the residues of the function

$$f(z) = \frac{\sin z^2}{z^2\left(z-\frac{\pi}{4}\right)}$$

at its singular points.

Solution. It is obvious that $z = 0$ and $z = \pi/4$ are the end singular points of the given function.

Let us find the limits of the function $f(z)$ at these points:

$$\lim_{z\to 0} f(z) = \lim_{z\to 0}\frac{\sin z^2}{z^2}\lim_{z\to 0}\frac{1}{z-(\pi/4)} = -\frac{4}{\pi},$$

$$\lim_{z\to \pi/4} f(z) = \infty.$$

Thus, $z = 0$ is a removable singular point and $\operatorname{Res}_{z=0} f(z) = 0$. Further, $z = \pi/4$ is a pole and

$$\operatorname{Res}_{z=\pi/4} f(z) = \lim_{z\to\pi/4} f(z)\left(z-\frac{\pi}{4}\right) = \lim_{z\to\pi/4}\frac{\sin z^2}{z^2} = \frac{16}{\pi^2}\sin\frac{\pi^2}{16}.$$

The point $z = \infty$ is also singular, it is an essential singular point. To find the residue of $f(z)$ at this point, one has to expand the function in a Laurent's series in powers of z:

$$f(z) = \frac{1}{z^3}\sum_{n=0}^{\infty}\left(\frac{\pi}{4z}\right)^n \cdot \sum_{k=0}^{\infty}(-1)^k\frac{z^{2(2k+1)}}{(2k+1)!}.$$

Multiplying the series and grouping the terms with like powers of z, we obtain

$$f(z) = \frac{1}{z}\left(1-\frac{\pi^4}{4^4 3!}+\frac{\pi^8}{4^8 5!}-\frac{\pi^{12}}{4^{12}7!}+\ldots\right)+\sum_{k\neq -1} c_k z^k,$$

i.e.

$$\operatorname{Res}_{z=\infty} f(z) = -\left(1-\frac{\pi^4}{4!\,3!}+\frac{\pi^8}{4^8 5!}-\ldots\right)$$

$$= -\frac{4^2}{\pi^2}\left(\frac{\pi^2}{4^2}-\frac{\pi^6}{4^6 3!}+\frac{\pi^{10}}{4^{10}5!}-\ldots\right) = -\frac{16}{\pi^2}\sin\frac{\pi^2}{16}.$$

The same result will be obtained if we take advantage of the basic theorem on residues according to which the residue of the function $f(z)$ with respect to $z = \infty$ is equal to the sum of the residues with respect to the end singular points taken with the minus sign (see [3], Sec. 6.13, Theorem 1).

Let us consider one more method for finding the residue of the function $f(z)$ at the point $z = \infty$. The function $\frac{1}{z^2}\sin z^2$ may be regarded as analytic on the z-plane, if it is supposed that it is equal

to 1 at $z = 0$. Therefore it is expanded into a power series in powers of $\left(z-\frac{\pi}{4}\right)$, which converges in the entire z-plane:

$$\frac{\sin z^2}{z^2} = \frac{\sin (\pi/4)^2}{(\pi/4)^2} + \sum_{k=1}^{\infty} a_k \left(z-\frac{\pi}{4}\right)^k.$$

But then

$$f(z) = \frac{1}{z-\frac{\pi}{4}} \cdot \frac{\sin (\pi/4)^2}{(\pi/4)^2} + \psi(z) = \frac{1}{z-\frac{\pi}{4}} \cdot \frac{\sin (\pi/4)^2}{(\pi/4)^2} + \sum_{k=0}^{\infty} c_k z^k,$$

where the power series on the right is convergent for all z. In fact, $\psi(z)$ is a function analytic in the z-plane and it is expanded in a power series convergent in this plane. Now, taking into consideration Problem 663, we obtain

$$\operatorname*{Res}_{z=\infty} f(z) = \operatorname*{Res}_{z=\infty} \frac{\sin (\pi/4)^2}{(\pi/4)^2} \cdot \frac{1}{z-\frac{\pi}{4}} = -\frac{\sin (\pi/4)^2}{(\pi/4)^2}.$$

665. Find the residues of the indicated functions at their singular points:

(a) $f(z) = z^2 \sin (1/z^2)$; (b) $f(z) = z^2 \exp (1/z)$;

(c) $f(z) = \dfrac{\exp (z)}{(z+1)^3 (z-2)}$; (d) $f(z) = (z-2) \exp \left(\dfrac{1}{z-2}\right)$.

Sec. 11.9. Computing Integrals with the Aid of Residues (See [3], Sec. 6.14)

The basic theorem on residues can also be formulated as follows: the integral of $f(z)$ about a contour Γ, traced anticlockwise, is equal to the sum of residues with respect to all singular points $z_1, \ldots, z_n$, found inside Γ, multiplied by $2\pi i$:

$$\int_{\Gamma} f(z)\, dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res} f(z_k). \tag{1}$$

666. Evaluate the integral

$$\int_{|z|=2} \frac{e^z-1}{z(z+1)}\, dz.$$

Solution. In the domain $|z| < 2$ the function $f(z) = \dfrac{e^z-1}{z(z+1)}$ is analytic everywhere except for the points $z = 0$, $z = -1$. Let us

find the residues of $f(z)$ at these points. Since $z = 0$ is a removable singular point, $\operatorname{Res}_{z=0} f(z) = 0$. At the point $z = -1$ the function $f(z)$ has a simple pole, therefore

$$\operatorname{Res}_{z=-1} f(z) = \lim_{z\to -1} (z+1) f(z) = \lim_{z\to -1} \frac{e^z - 1}{z} = 1 - e^{-1}.$$

According to (1), we obtain $\int\limits_{|z|=2} \frac{e^z-1}{z(z+1)}\, dz = 2\pi i(1-e^{-1})$.

667. Evaluate the given integrals:

(a) $\int\limits_{\Gamma} \frac{z\, dz}{(z-1)^2(z-3)}$, where Γ: $\frac{x^2}{4} + \frac{y^2}{9} = 1$;

(b) $\int\limits_{|z|=5} \frac{e^{iz}\, dz}{(z-\pi)^3}$; (c) $\int\limits_{\Gamma} \frac{dz}{z^4+1}$, Γ: $x^2+y^2 = 2x$;

(d) $\int\limits_{\Gamma} \frac{z+1}{z^2+2z-3}\, dz$, where Γ: $\frac{x^2}{4} + y^2 = 1$.

668. Compute the integral

$$I = \int\limits_{-\infty}^{\infty} \frac{x^2\, dx}{(x^2+a^2)^2} \quad (a > 0).$$

Solution. We introduce the function $f(z) = \frac{z^2}{(z^2+a^2)^2}$ of the complex variable z. It satisfies the conditions of Theorem 1 from Sec. 6.14 [3] for $m = 4$ and has in the upper half-plane a pole of the second order at the point $z = ai$:

$$\operatorname{Res}_{z=ai} f(z) = \lim_{z\to ai} \frac{d}{dz}(z-ai)^2 f(z) = \lim_{z\to ai} \frac{d}{dz} \frac{z^2}{(z+ai)^2} = \frac{1}{4ai}.$$

Then

$$I = 2\pi i \operatorname{Res}_{z=ai} f(z) = \frac{\pi}{2a}.$$

669. Evaluate the integrals:

(a) $\int\limits_0^{\infty} \frac{x^2+1}{x^4+1}\, dx$; (b) $\int\limits_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$ $(a > 0,\ b > 0)$;

(c) $\int\limits_{-\infty}^{\infty} \frac{dx}{1+x^6}$; (d) $\int\limits_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}}$ $(n = 1, 2, \ldots)$.

670. Evaluate the following integrals:

(a) $\int\limits_0^\infty \frac{x \sin x}{x^2+a^2}\,dx \quad (a>0)$; (b) $\int\limits_{-\infty}^{\infty} \frac{x^2 \cos x}{(x^2+1)^2}\,dx$.

671. Compute the integral

$$I = \int\limits_0^\infty \frac{\sin x}{x(x^2+a^2)}\,dx \quad (a>0).$$

Hint. Consider the function $f(z) = \frac{e^{iz}}{z(z^2+a^2)}$ and the contour $\Gamma_{\varepsilon, R}$ (Fig. 41). For a small ε and large R the function $f(z)$ has one pole inside the contour $\Gamma_{\varepsilon, R}$. Note that $f(z)$ has a singularity at the point $z = 0$ on the real axis. Then pass to the limit:

$$\lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \int\limits_{\Gamma_{\varepsilon, R}} f(z)\,dz.$$

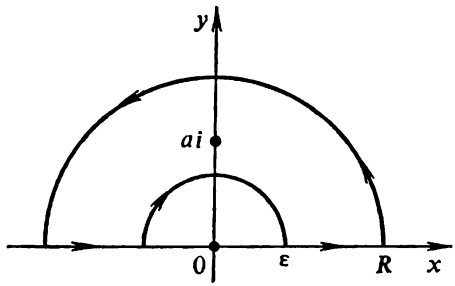

Fig. 41.

672. Compute the integral

$$I = \int\limits_0^\infty \exp(-ax^2) \cos bx\,dx \quad (a>0,\ b>0).$$

Solution. As we know,

$$\int\limits_{-\infty}^{\infty} \exp(-x^2)\,dx = \sqrt{\pi},$$

therefore

$$\int\limits_{-\infty}^{\infty} \exp(-ax^2)\,dx = \sqrt{\frac{\pi}{a}}.$$

Consider the function $f(z) = \exp(-az^2)$ and the contour Γ having the shape of a rectangle with sides $2R$ and $b/(2a)$ (Fig. 42). Inside the contour Γ the function $\exp(-az^2)$ is analytic, therefore

$$\int_{\Gamma} \exp(-az^2)\, dz = 0,$$

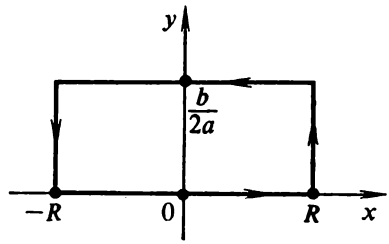

Fig. 42.

i.e.

$$\int_{-R}^{R} \exp(-ax^2)\, dx + \int_{0}^{b/2a} i \exp[-a(R+iy)^2]\, dy + \int_{R}^{-R} \exp\left[-a\left(x+i\frac{b}{2a}\right)^2\right] dx + \int_{b/2a}^{0} i \exp[-a(-R+iy)^2]\, dy = 0.$$

As $R \to \infty$, the second and fourth integrals tend to zero at the expense of the factor $\exp(-aR^2)$. Therefore, in the limit, as $R \to \infty$, we obtain

$$\int_{-\infty}^{\infty} \exp(-ax^2)\, dx + \int_{\infty}^{-\infty} \exp(-ax^2) \exp(-ibx) \exp\frac{b^2}{4a}\, dx = 0.$$

Hence, separating the real part in the second term, we have

$$\exp\left(\frac{b^2}{4a}\right) \int_{-\infty}^{\infty} \exp(-ax^2) \cos bx\, dx = \int_{-\infty}^{\infty} \exp(-ax^2)\, dx = \sqrt{\frac{\pi}{a}}$$

or

$$\int_{0}^{\infty} \exp(-ax^2) \cos bx\, dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} \exp\left(-\frac{b^2}{4a}\right).$$

Chapter 12

OPERATIONAL CALCULUS
(See [3], Chapter 7)

Sec. 12.1. Transforms of Simplest Functions (See [3], Secs. 7.1 and 7.2)

673. Using the definition, find the Laplace transforms of the indicated functions:

(a) $f(t) = e^{2t}$; (b) $f(t) = \sin 3t$.

674. Is it possible for the function $\varphi(p) = 1/\sin p$ to be the transform of some object (or original) function?

675. Using the property of linearity and the property of similarity find the transforms of the following functions:

(a) $f(t) = t+2$; (b) $f(t) = 2\sin 3t + e^{-2t}$ $(t \geqslant 0)$.

676. Is

$$f(t) = \begin{cases} \exp(t^2), & t \geqslant 0, \\ 0, & t < 0, \end{cases}$$

an object function?

677. Find the transform of the function $f(t) = \cos mt \cos nt$.

678. Using the theorem on differentiation of object function

$$L[f'; p] = pL[f; p] - f(0),$$

find the transforms of the following functions:

(a) $f(t) = \cos^2 3t$; (b) $f(t) = \cos^4 t$.

679. Using the theorem on differentiation of result (or image) function $(F'(p) \fallingdotseq -tf(t))$, find the transforms of the below functions:

(a) $f(t) = t^2 \cos t$; (b) $f(t) = t \sinh 3t$.

680. Using the theorem on integration of object function

$$\int_0^t f(\tau)\, d\tau \fallingdotseq \frac{F(p)}{p},$$

find the transforms of the following functions:

(a) $f(t) = \int_0^t \tau \sinh 3\tau\, d\tau$; (b) $f(t) = \int_0^t \tau^2 \cos \tau\, d\tau$.

681. Using the theorem on integration of result function

$$\frac{f(t)}{t} \doteq \int_p^{\infty} F(q)\,dq,$$

find the transforms of the following functions:

(a) $\frac{e^t-1}{t}$; (b) $\frac{\sin^2 t}{t}$; (c) $\frac{\sinh t}{t}$.

682. Using the theorem on translation (shift) of result function, find the transforms of the indicated functions:

(a) $e^{3t}\sin t$, (b) $e^t t^2 \cos t$.

683. Using the theorem on delay of object function

$$f(t-t_0) \doteq e^{-pt_0}F(p),$$

find the transforms of the below functions:

(a) $\sin(t-b)\,\sigma_0(t-b)$;

(b) $e^{t-3}\sigma_0(t-3)$, where $\sigma_0(t) = \begin{cases} 1, & t \geqslant 0, \\ 0, & t < 0. \end{cases}$

684. Find the transforms of the following functions:

(a) $f(t) = \begin{cases} t, & 0 \leqslant t \leqslant 1, \\ 2-t, & 1 \leqslant t \leqslant 2, \end{cases}$ $f(t+2) = f(t)$, $\forall\, t \geqslant 0$;

(b) $f(t) = |\sin t|$.

685. Find the transform of the given convolutions:

(a) $\int_0^t e^{t-\tau}\sin\tau\,d\tau$; (b) $\int_0^t (t-\tau)^2\cosh\tau\,d\tau$.

Sec. 12.2. Finding Object Function by Its Transform (See [3], Sec. 7.2)

686. Find the object function $f(t)$ if $F(p) = 1-\cos(1/p)$.

687. Find the object function for the result function

$$F(p) = \frac{1}{p(p-1)(p^2+1)}.$$

688. Find the object function for the indicated result function:

(a) $F(p) = \frac{1}{p^2+4p+3}$; (b) $F(p) = \frac{1}{p^2(p^2+1)}$.

689. Find the object function for the function $F(p) = 1/\sqrt{1+p^2}$.

Solution. Let us expand the result function $F(p)$ into a Laurent's series in the neighbourhood of a point at infinity:

$$F(p) = \frac{1}{p}\left(1+\frac{1}{p^2}\right)^{-1/2} = \frac{1}{p}\left(1-\frac{1}{2p^2}+\frac{\frac{1}{2}\cdot\frac{3}{2}}{2!\,p^4}-\ldots\right)$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k(2k)!}{(k!)^2\,2^{2k}p^{2k+1}}.$$

By Theorem 11 (see [3], Sec. 7.2),

$$f(t) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(k!)^2\,2^{2k}}\cdot t^{2k} \equiv J_0(t),$$

where $J_0(t)$ is a Bessel's function of the zero order (see [3], Secs. 1.25 and 5.9).

690. Find the object function for the rational functions $F(p)$, using the equality

$$f(t) = \sum_{k=1}^{m}\operatorname*{Res}_{p=p_k}[F(p)e^{pt}],$$

where $p_1, \ldots, p_m$ are the poles of the function $F(p)$:

(a) $F(p) = \dfrac{1}{p^2+4p+5}$; (b) $F(p) = \dfrac{2p^3+p^2+2p+2}{p^3(p^2+2p+2)}$.

Sec. 12.3. Applications of Operational Calculus (See [3], Sec. 7.3)

691. Solve the given differential equations for the indicated initial conditions:

(a) $x'+x = e^{-t}$, $x(0) = 1$;

(b) $x''+x = 2\cos t$, $x(0) = 0$, $x'(0) = -1$;

(c) $x''+2x'+5x = 3$, $x(0) = 1$, $x'(0) = 0$.

692. With the aid of Duhamel's formula, solve the following equations:

(a) $x'' = \dfrac{1}{1+t^2}$, $x(0) = x'(0) = 0$;

(b) $y''+y = \sin x$, $y(0) = y'(0) = 0$;

(c) $y'''+y' = 10e^{2x}$, $y(0) = y'(0) = y''(0) = 0$.

693. Solve the system

$$\begin{cases} x'+y=0, \\ y'+x=0, \end{cases} \quad x(0)=1, \quad y(0)=-1.$$

Solution. Let $X(p) \fallingdotseq x(t)$, $Y(p) \fallingdotseq y(t)$. Write the operator equations

$$\begin{cases} pX(p)-x(0)+Y(p)=0, \\ X(p)+pY(p)-y(0)=0, \end{cases}$$

or

$$\begin{cases} pX(p)+Y(p)=1, \\ X(p)+pY(p)=-1. \end{cases}$$

Solving this system with respect to $X(p)$ and $Y(p)$, we have

$$X(p)=\frac{1}{p-1}, \quad Y(p)=\frac{-1}{p-1}.$$

Hence,

$$x(t)=e^t, \quad y(t)=-e^t.$$

694. Solve the following systems:

(a) $\begin{cases} x+x'=y+e^t, \\ y+y'=x+e^t, \end{cases} \quad x(0)=y(0)=1;$

(b) $\begin{cases} x''=3(y-x+z), \\ y''=x-y, \\ z''=-z, \end{cases}$

$x(0)=x'(0)=0; \quad y(0)=0, \quad y'(0)=-1; \quad z(0)=1, \quad z'(0)=0;$

(c) $\begin{cases} y'=3z-y, \\ z'=y+z+e^x, \end{cases} \quad y(0)=z(0)=0.$

695. Compute the below integrals:

(a) $I(x)=\int\limits_0^\infty \frac{\cos xt}{a^2+t^2}\,dt;$ (b) $I(x)=\int\limits_0^\infty \frac{\sin xt \cos t}{t}\,dt.$

8. $a = 0.(1) = 1/9$. **9.** $a+b = 0.(25) = 25/99$. **10.** $A+B = [2, 6)$, $AB = (3, 5]$, $A\setminus B = [2, 3]$. **11.** (a) $-3.1 < x < -2.9$; (b) $x \leqslant -7$, $x \geqslant 13$; (c) $x < -3/2$; (d) $0 < x < 1/2$; (e) $x \leqslant -3/2$, $x \geqslant 1/4$. **12.** $a > -a$, if $a > 0$, $-a > a$, if $a < 0$. **13.** (a) $b \geqslant 0$; (b) $b \leqslant 0$; (c) $b < 0$, (d) $b > 0$. **14.** (a) 1; (b) 1; (c) $\sqrt{2}/2$. **16** 0. **17.** 0. **18.** 1/2 (see Problem 1). **19.** 1/3 (see Problem 2). **22.** No. It is simply unbounded. **27.** 9/8; 1/6. **28.** 2; -30. **29.** 0, 1, 1, 1; $-1/6$, 4/3, 1/3, 1. **30.** -1, 0, 1. **32.** *Hint.* Take advantage of the inequality $\frac{1}{k^2} > \frac{1}{k-1} - \frac{1}{k}$ $(k \geqslant 2)$. **34.** $E = (-\infty, \infty)$; $E_1 = (0, 1]$.

35. $E = \left[\frac{3-\sqrt{17}}{2}, \frac{3+\sqrt{17}}{2}\right]$, $E_1 = \left[0, \frac{\sqrt{17}}{2}\right]$.

36. $1, -\frac{x^2+4x+3}{x^2+4x+5}$, $-f(x)$, $2/(1+x^2)$, $(1+x^2)/(1-x^2)$.

42. 0, 25; 2, ∞. **43.** $f(x) = \begin{cases} \sin x, & 0 \leqslant x \leqslant \pi/2, \\ 1, & x > \pi/2 \end{cases}$ (Fig. 43);

$$\varphi(x) = \begin{cases} 0, & 0 \leqslant x \leqslant \pi, \\ \sin x, & \pi < x \leqslant 3\pi/2, \\ -1, & x > 3\pi/2 \end{cases}$$ (Fig. 44).

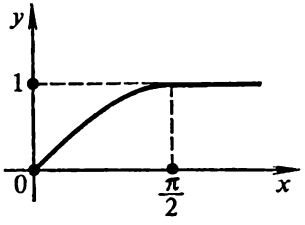

Fig. 43.

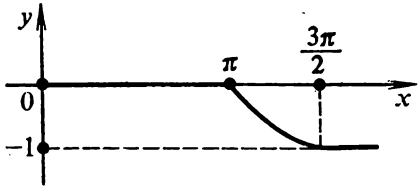

Fig. 44.

44. (a) 1; (b) 2/3; (c) 1/2; (d) 3; (e) 4/3; (f) 0; (g) $-1/16$, (h) 1/144, (i) $1/\sqrt{2a}$; (j) 0. **45.** (a) 3; (b) 2; (c) 4. **46.** ∞. **47.** (a) e^4; (b) e^3; (c) 1. **48.** (a) 4; (b) $a^b \ln a$. **49.** Continuous everywhere (Fig. 45). **50.** If $A = 2$, then $f(x)$ is continuous everywhere. If $A \neq 2$, then $x = 1$ is a removable discontinuity (Fig. 46). **51.** $x = -1$, $x = 3$ are points of discontinuity of the first kind (Fig. 47). **52.** $x = -1$ is a removable discontinuity (Fig. 48). **53.** $x = -1$ is a point of discontinuity of the second kind.

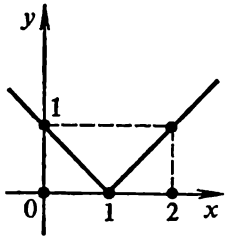

Fig. 45.

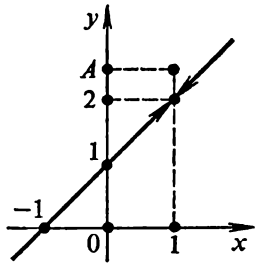

Fig. 46.

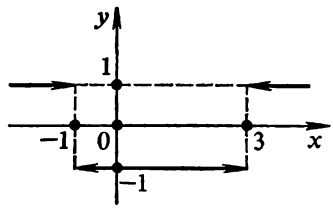

Fig. 47.

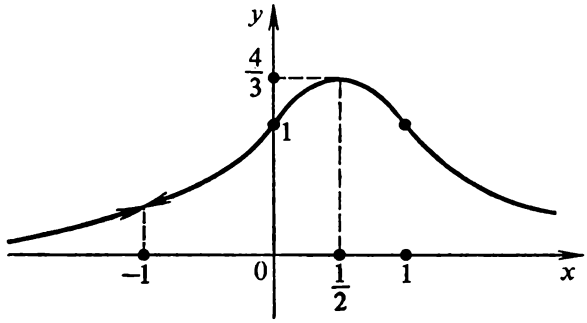

Fig. 48.

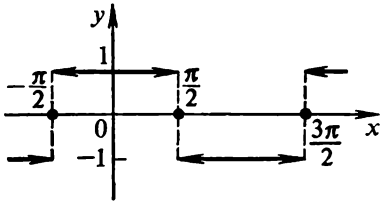

Fig. 49.

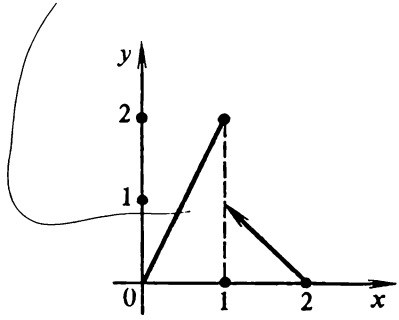

Fig. 50.

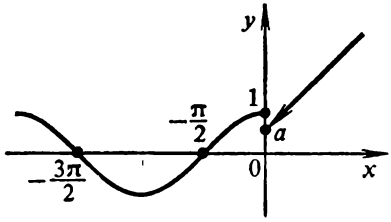

Fig. 51.

54. $x = \dfrac{2k+1}{2}\pi$ $(k = 0, \pm 1, \ldots)$ are points of discontinuity of the first kind (Fig. 49). **55.** $x = 1$ is a point of discontinuity of the first kind (Fig. 50). **56.** $x = 0$ is a point of discontinuity of the first kind if $a \neq 1$ (Fig. 51). **57.** $\delta \leqslant \varepsilon/8$. **58.** $\delta \leqslant \varepsilon/12$. **59.** $(b-dy)/(cy-a)$. **60.** $3x$. **61.** x. **62.** $f' = 3x^2-2$, $f'(0) = -2$, $f'(2) = 10$.

63. $f'(x) = \arcsin\dfrac{x}{x+1} + \dfrac{x}{(x+1)\sqrt{2x+1}}$ $\left(x > -\dfrac{1}{2}\right)$; $f'(0) = 0, f'(1) = \dfrac{\pi}{6} + \dfrac{1}{2\sqrt{3}}$. **64.** $\dfrac{2(1+x^2)}{(1-x^2)^2}$ $(|x| \neq 1)$.

65. (a) $1 + \dfrac{1}{3\sqrt[3]{x^2}}$ $(x \neq 0)$; (b) $-\dfrac{1}{x^2} - \dfrac{1}{2x\sqrt{x}} - \dfrac{1}{3x\sqrt[3]{x}}$ $(x > 0)$;

(c) $\dfrac{1-x+4x^2}{(1-x)^3(1+x)^4}$ $(|x| \neq 1)$; (d) $\dfrac{1+2x^2}{(1+x^2)^{1/2}}$;

(e) $\dfrac{a^2}{(a^2-x^2)^{3/2}}$ $(|x| < |a|)$;

(f) $\dfrac{1+2\sqrt{x}+4\sqrt{x}\sqrt{x+\sqrt{x}}}{8\sqrt{x}\sqrt{x+\sqrt{x}}\sqrt{x+\sqrt{x+\sqrt{x}}}}$ $(x > 0)$;

(g) $\dfrac{2}{3\sqrt[3]{x}} + \dfrac{1}{x\sqrt{x}}$ $(x > 0)$; (h) $\dfrac{2}{\sin^2 x}$ $(x \neq k\pi, \quad k = 0, \pm 1, \ldots)$;

(i) $\dfrac{x^2}{(\cos x + x \sin x)^2}$; (j) $\dfrac{n \sin x}{\cos^{n+1} x}$ $\left(x \neq \dfrac{2k-1}{2}\pi, \quad k-\text{integer}\right)$;

(k) $-2^{\tan\frac{1}{x}}\left(\cos\dfrac{1}{x}\right)^{-2}\dfrac{\ln 2}{x^2}$; (l) $e^x e^{e^x}(1 + e^{e^x})$;

(m) $a^{a^x}x^{a^a-1} + ax^{a-1}a^{x^a}\ln a$; (n) $\dfrac{-2 \operatorname{sgn} x}{1+x^2}$ $(x \neq 0)$;

(o) $\dfrac{\operatorname{sgn} x}{\sqrt{1-x^2}}$ $(0 < |x| < 1)$.

66. $\dfrac{1}{\cos^4 x}$ $\left(x \neq \dfrac{2k+1}{2}\pi, \quad k = 0, \pm 1, \ldots\right)$. **67.** $-2x \exp(-x^2)$.

68. (a) $2x \cos x^2$; (b) $\sin 2x$; (c) $21x^6 \cos x^7 \sin^2 x^7$; (d) $-\cos x \cdot \sin(\sin x)$; (e) $-2x \sin x^2$; (f) $-4x^3 \sin 2x^4$. **69.** (a) $1/\sqrt{a^2-x^2}$ $(a > 0)$; (b) $a/(a^2+x^2)$; (c) $1/\sqrt{a^2+x^2}$; (d) $\operatorname{sgn}\cos x$, $x \neq \dfrac{2k+1}{2}\pi$, $k-$integer; (e) $-1/\sqrt{a^2-x^2}$ $(a > 0)$; (f) $(3x^2+1)e^{x^3+x}$. **70.** $\dfrac{1}{\sin x}$ $(0 < x - 2k\pi < \pi, \quad k-\text{integer})$.

71. (a) $\dfrac{x}{1+x^2} + \arctan x$; (b) $\dfrac{6}{x}\ln^2 x^2$ $(x \neq 0)$; (c) $\dfrac{1}{x \ln x \ln(\ln x)}$ $(x > e)$;

(d) $\dfrac{x}{x^4-1}$ $(|x| > 1)$; (e) $\dfrac{1}{2(1+\sqrt{x+1})}$ $(x > -1)$; (f) $1/\cos x \times$ $\times(|x - 2k\pi| < \pi/2, \quad k-\text{integer})$; (g) $-\dfrac{\ln^3 x}{x^2}$ $(x > 0)$; (h) $\dfrac{2x}{1+\sqrt[3]{1+x^2}}$;

(i) $\dfrac{\sqrt{x}}{2(1+x)}$ $(x \geqslant 0)$; (j) $\dfrac{1}{1+x^2}$ $(x \neq 1)$; (k) $\dfrac{x^3}{x^6+1}$ $\left(|x| \neq \dfrac{1}{\sqrt{2}}\right)$;
(l) $\dfrac{12x^5}{(1+x^{12})^2}$; (m) $\dfrac{\sin 2x}{\sin^4 x+\cos^4 x}$ $\left(x \neq \dfrac{2k-1}{2}\pi,\ k\text{—integer}\right)$;
(n) $\dfrac{x^2}{1+x^2}\arctan x$; (o) $\dfrac{1}{2(1+x^2)}$.

72. $f'(x) = \begin{cases} -1, & -2 < x < 1, \\ 2x-3, & 1 \leqslant x \leqslant 2, \\ 1, & 2 < x < 4 \end{cases}$ (Fig. 52).

73. $f'(x) = \begin{cases} 1, & x < 0, \\ \dfrac{1}{1+x}, & x \geqslant 0 \end{cases}$ (Fig. 53).

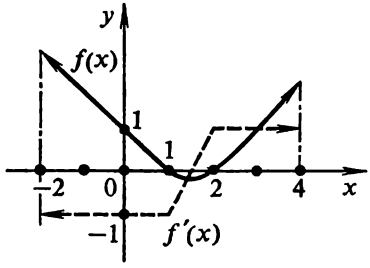

Fig. 52.

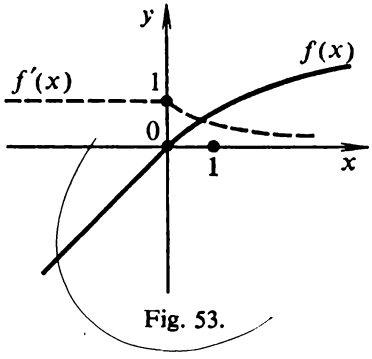

Fig. 53.

74. $\dfrac{y'}{y} = \dfrac{1-x-x^2}{x(1-x^2)}$. 75. $2\tanh x$. 76. (a) $2x\cosh(x^2+1)$;
(b) $18x^5\sinh^2 x^6\cosh x^6$. 77. $(2x+1)\sinh 2(x^2+x+1)$. 78. (a) $2\tanh x/\cosh^2 x$;
(b) $2x/\cosh^2 x^2$. 79. (a) $\dfrac{1}{x\cosh^2(1+\ln x)}$; (b) $1/\sqrt{1+x^2}$; (c) $\dfrac{\sqrt{x+\sqrt{1+x^2}}}{\sqrt{2}(1+x^2)^{3/4}}$;
(d) $\coth x$ $(x > 0)$; (e) $\dfrac{1}{x}\sinh\ln x$ $(x > 0)$; (f) $\dfrac{1}{\cosh^2 x}e^{\tanh x}$;
(g) $(\sinh x)^{\cosh x}[\sinh x\ln\sinh x+\cosh x\coth x]$; (h) $-(3\sinh x)/\cosh^4 x$;
(i) $2\cosh x/\sinh^2 x$ $(x \neq 0)$. 80. $df(x) = (2x+1)\Delta x$; $df(1) = 3\Delta x$; $\Delta f(1) = 3\Delta x+(\Delta x)^2$; for $\Delta x = 0.1$, $df(1) = 0.3$, $\Delta f(1) = 0.31$. 81. $e^x(1+x)\,dx$.
82. $\cosh x\,dx$. 83. $-x\sinh x\,dx$. 84. $-\dfrac{2x\,dx}{1-x^2}$ $(|x| < 1)$.
85. (a) $(-2+4x^2)e^{-x^2}$; (b) $\dfrac{x(3+2x^2)}{(1+x^2)^{3/2}}$. 87. $120x\,dx^4$. 88. $e^x\left[\ln x+\dfrac{3}{x}-\dfrac{3}{x^2}+\dfrac{2}{x^3}\right]dx^3$.
89. $y' = \dfrac{3-3t^2}{2-2t} = \dfrac{3}{2}(1+t)$, $y'' = \dfrac{3}{4(1-t)}$ $(t \neq 1)$. 90. $y' = -\cot t$, $y'' = -1/(2\sin^3 t)$ $(t \neq k\pi,\ k\text{—integer})$. 91. $y' = t$, $y'' = 1/f''(t)$. 92. $y' = \dfrac{\sin t}{1-\cos t}$, $y'' = \dfrac{-1}{4\sin^4(t/2)}$ $(t \neq 2k\pi,\ k\text{—integer})$. 93. (a) $y+11x-18 = 0$; $11y-x+46 = 0$; (b) have no common tangent. 94. $\varphi = \arctan 2\sqrt{2}$.

95. $\varphi = \pi/4$. **96.** $\pi/6$. **98.** (a) $c = 1/\sqrt{3}$; (b) $c = \dfrac{-A \pm \sqrt{A^2+3A+3}}{3}$, $\forall B, C$; (c) $c = 1/\sqrt{3}$, $\forall A, B, C$. **100.** (a) The function is increasing on $(-\infty, 1/2)$ and decreasing on $(1/2, \infty)$; (b) Increasing on $(-\infty, 1)$, and decreasing on $(1, \infty)$. **103.** a/b. **104.** 2. **105.** 1. **106.** 4. **107.** 1/4. **108.** 1. **109.** (a) $e^{-1/6}$; (b) $e^{1/3}$; (c) a/b; (d) 3; (e) $-1/3$; (f) 2; (g) 1/3; (h) 0; (i) e^{-1}; (j) 1; (k) 1; (l) e^{-1}; (m) $a^a(-1+\ln a)$; (n) $1/n$; (o) $\dfrac{\alpha}{m} - \dfrac{\beta}{n}$; (p) n/m; (q) 1/2; (r) $2/\pi$; (s) 1/2. **111.** $\tan x = x + \dfrac{x^3}{3} + \dfrac{2}{15}x^5 + o(x^6)$. **112.** $f(x) = 1 + 2x + x^2 - \dfrac{2}{3}x^3 + o(x^4)$. *Hint.* Use the expansion by Taylor's formula for the function e^x. **113.** $\ln\cos x = -\dfrac{x^2}{2} - \dfrac{x^4}{12} + o(x^5)$. **114.** $\sin(\sin x) = x - \dfrac{x^3}{3} + o(x^4)$. **115.** $-1/12$. **116.** 0. **117.** $-1/6$. **118.** $-1/2$. **119.** (a) 2.718281; (b) 0.017453; (c) 2.236. *Hint.* Represent the number $\sqrt{5}$ in the form $\sqrt{5} = 2\sqrt{1+\dfrac{1}{4}}$ and take advantage of Taylor's formula for the function $(1+x)^m$ for $x = 1/4$, $m = 1/2$; (d) $\ln 2 = 0.69315$, $\ln 3 = 1.09861$. **120.** (a) For $x = -1/2$ a maximum, $y(-1/2) = 9/4$; (b) for $x = 0$ a minimum, $y(0) = 0$, (c) for $x = 0$ a minimum, $y(0) = 0$; for $x = \pm 1$ a maximum, $y(\pm 1) = 1$; (d) for $x = -1$ a maximum, $y(-1) = -2$; for $x = 1$ a minimum, $y(1) = 2$; (e) for $x = 3\pi/4$ a maximum, $y\left(\dfrac{3\pi}{4}\right) = \dfrac{\sqrt{2}}{2}e^{3\pi/4}$; for $x = \dfrac{7\pi}{4}$ a minimum, $y\left(\dfrac{7\pi}{4}\right) = -\dfrac{\sqrt{2}}{2}e^{7\pi/4}$; (f) for $x = 0$ a maximum, $y(0) = 1/4$; (g) for $x = 1/2$ a maximum, $y(1/2) = 1/4$; for $x = -1/2$ a minimum, $y(-1/2) = -1/4$; (h) for $x = 1$ a maximum, $y(1) = 0$; for $x = 3$ a minimum, $y(3) = -4$; (i) for $x = k\pi$ ($k = 0, \pm 1, \ldots$) a maximum, $y(k\pi) = (-1)^k + \dfrac{1}{2}$; for $x = \pm\dfrac{2\pi}{3} + 2k\pi$ ($k = 0, \pm 1, \ldots$) a minimum, $y\left(\pm\dfrac{2\pi}{3} + 2k\pi\right) = -\dfrac{3}{4}$; (j) for $x = \dfrac{\pi}{3} + 2k\pi$ ($k = 0, \pm 1, \ldots$) a maximum, $y\left(\dfrac{\pi}{3} + 2k\pi\right) = \dfrac{3\sqrt{3}}{4}$; for $x = -\dfrac{\pi}{3} + 2k\pi$ ($k = 0, \pm 1, \ldots$) a minimum, $y\left(-\dfrac{\pi}{3} + 2k\pi\right) = -\dfrac{3\sqrt{3}}{4}$. **121.** (a) 32; 1; (b) 10, 1; 2. **122.** $d = 7\sqrt{2}/8$. *Hint.* Investigate the function $y = \dfrac{1}{\sqrt{2}}|x^2 - x + 2|$ for extremum. **123.** (a) 2; 0; (b) $(1+\sqrt{2})/2 - 1$; 0. **124.** (a) $x = 1$ is a point of inflection; the graph is convex downward on $(-\infty, 1)$; on $(1, \infty)$ the graph is convex upward; (b) $x = \pm 1/\sqrt{2}$ are inflection points; on $(-1/\sqrt{2}, 1/\sqrt{2})$ the graph is convex upward; on the intervals $|x| > 1/2$ the graph is convex downward (Fig. 54). **125.** (a) Inclined asymptote: $y = x$; vertical asymptote: $x = 0$ (Fig. 55); (b) horizontal asymptote: $y = 1$, $x = 0$—a vertical asymptote as $x \to +0$ (Fig. 56); (c) $y = 0$—a horizontal asymptote (see Fig. 54), (d) $y = x$—an inclined asymptote as $x \to \infty$; horizontal asymptote: $y = 0$ as $x \to -\infty$ (Fig. 57).

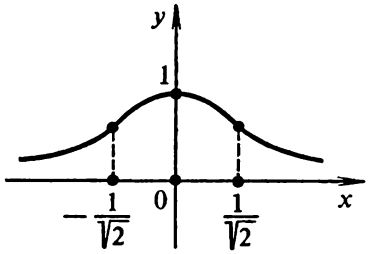

Fig. 54.

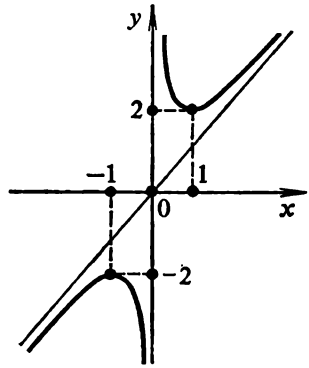

Fig. 55.

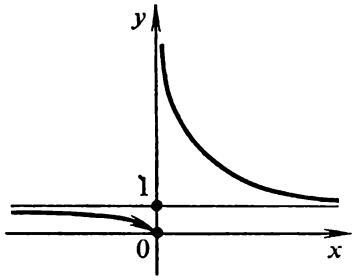

Fig. 56.

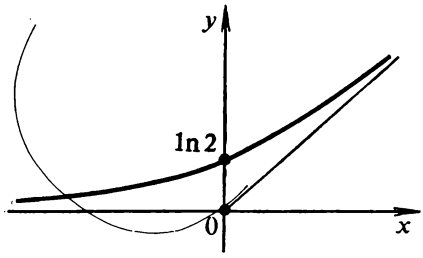

Fig. 57.

126. (a) $x = -1/3$—a point of minimum, $y(-1/3) = -\sqrt{10}$; $x = -1$, $x = 1/2$ are points of inflection; $y = -1$ is a horizontal asymptote as $x \to -\infty$ and $y = 1$ is the same for $x \to +\infty$ (Fig. 58); (b) $x = 0$ is a point of minimum; $x = -1$ is a vertical asymptote; $y = 0$ is a horizontal asymptote as $x \to -\infty$; the graph is convex upward on $(-\infty, -1)$ and downward on $(-1, \infty)$ (Fig. 59). **127.** (a) See Fig. 60, Γ_1 $(t > 1)$, Γ_2 $(-1 < t < 1)$, Γ_3 $(t < -1)$; (b) See Fig. 61, Γ_1 $(t > 0)$, Γ_2 $(t < 0)$. **128.** (a) The sides of the rectangle: $a\sqrt{2}$ and $b\sqrt{2}$; (b) $x = a/6$; (c) $PB = 12$ km; (d) $a = d/\sqrt{3}$, $h = d\sqrt{2}/\sqrt{3}$. **129.** The rectangle is a square with side 2.

130. $\dfrac{|av \pm ub| \sin \theta}{\sqrt{u^2+v^2-2uv\cos\theta}}$. *Hint.* By the cosine law, the distance between the ships (see Fig. 7) at an arbitrary instant t is $r^2(t) = (a+ut)^2 + (b+vt)^2 - 2(a+ut)(b+vt)\cos\theta$. The instant of time when the ships were at a distance a and b from the meeting place is supposed to be equal to zero. Then investigate the function $r^2(t)$ for extremum. **131.** (a) $p\left(1+\dfrac{2x}{p}\right)^{3/2}$; (b) $\dfrac{(a^4-a^2x^2+b^2x^2)^{3/2}}{a^4b}$; (c) $2\sqrt{y}$. **132.** $\xi = a(t - 3\sin t)$, $\eta = 3a(1-\cos t)$.

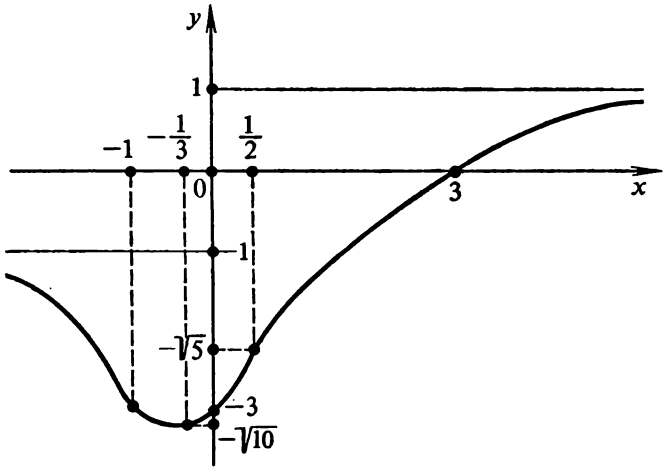

Fig. 58.

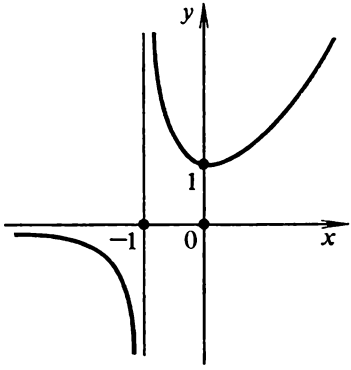

Fig. 59.

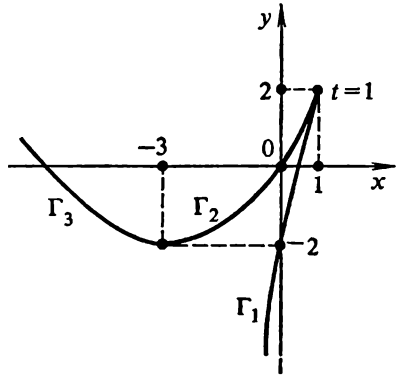

Fig. 60.

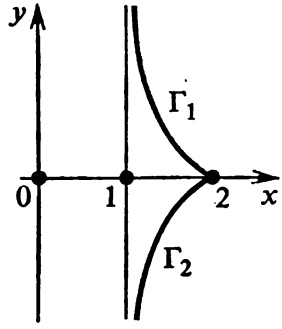

Fig. 61.

Chapter 2.

(For the sake of brevity, the constant C is omitted).

133. $\frac{625}{3}x^3-125x^4+30x^5-\frac{10}{3}x^6+\frac{1}{7}x^7$. **134.** $x-\frac{2}{3}x^3+\frac{1}{5}x^5$.

135. $x-\arctan x$. **136.** $x-\cos x+\sin x$. **137.** $x+\ln\left|\frac{1-x}{1+x}\right|$. **138.** $-x+\tan x$.

139. $x-\tanh x$. **140.** $x-\coth x$. **141.** $\operatorname{arc\,sin} x+\ln|x+\sqrt{1+x^2}|$.

142. $\frac{1}{200}(2x-3)^{100}$. **143.** (a) $\frac{1}{\sqrt{6}}\arctan\sqrt{\frac{3}{2}}x$; (b) $x-3x^2+\frac{11}{3}x^3-\frac{3}{2}x^4$;

(c) $a\ln|x|-\frac{a^2}{x}-\frac{a^3}{2x^2}$; (d) $\frac{4}{5}x\sqrt[4]{x}-\frac{24}{17}x\sqrt[12]{x^5}+\frac{4}{3}\sqrt[4]{x^3}$; (e) $-x+$

$+3\ln|x|+\frac{3}{x}-\frac{1}{2x^2}$; (f) $\frac{2}{3}\sqrt{x}(x+6)$; (g) $\frac{1}{4}\ln(1+x^4)$; (h) $-x+$

$+\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|$; (i) $x+2\ln\left|\frac{x-1}{x+1}\right|$; (j) $\frac{4^x}{\ln 4}+2\frac{6^x}{\ln 6}+\frac{9^x}{\ln 9}$; (h) $a\cosh x+$

$+b\sinh x$; (l) $\frac{1}{22}(2x-9)^{11}$; (m) $\frac{1}{\sqrt{3}}\operatorname{arc\,sin}\left(x\sqrt{\frac{3}{2}}\right)$;

(n) $\frac{1}{\sqrt{3}}\ln(\sqrt{3}x+\sqrt{2+3x^2})$; (o) $\frac{1}{\sqrt{3}}\ln|x\sqrt{3}+\sqrt{3x^2-2}|$; (p) $-\frac{1}{6}\times$

$\times(3e^{-2x}+2e^{-3x})$. **144.** $-\sqrt{1-x^2}$. **145.** $\ln(2+e^x)$. **146.** $\tan x-\cot x$.

147. $\frac{1}{3}[(x+1)^{3/2}-(x-1)^{3/2}]$. **148.** $-\ln|\cos x|$. **149.** $2\arctan\sqrt{x}$.

150. $-\frac{8+30x}{375}(2-5x)^{3/2}$. **151.** $\frac{1}{2(\ln 3-\ln 2)}\ln\left|\frac{3^x-2^x}{3^x+2^x}\right|$. *Hint.* Divide the numerator and denominator by 2^x3^x and set $t=(3/2)^x$. **152.** (a) $\frac{x}{2}\sqrt{x^2-2}+$

$+\ln|x+\sqrt{x^2-2}|$; (b) $\frac{1}{4}(1+x^3)^{4/3}$; (c) $\frac{-1}{2(1+x^2)}$; (d) $\frac{1}{8\sqrt{2}}\ln\left|\frac{x^4-\sqrt{2}}{x^4+\sqrt{2}}\right|$;

(e) $\arctan e^x$; (f) $x-\ln(1+\sqrt{e^{2x}+1})$; (g) $\frac{1}{3}\ln^3 x$; (h) $\ln|\sin x|$;

(i) $\ln|\tanh(x/2)|$; (j) $2\arctan e^x$; (k) $\frac{1}{2}(\arctan x)^2$; (l) $-\frac{x}{2}+\frac{1}{4}\sinh 2x$;

(m) $\frac{x}{2}+\frac{1}{4}\sinh 2x$; (n) $-2\coth 2x$; (o) $-\frac{1+2x}{10}(1-3x)^{2/3}$; (p) $\frac{3}{56}\times$

$\times(1+x^2)^{4/3}(4x^2-3)$; (q) $x-\ln(1+e^x)$. **153.** $x\arctan x-\frac{1}{2}\ln(1+x^2)$.

154. $-(x+1)e^{-x}$. **155.** $x\cosh x-\sinh x$. **156.** $-x\cos x+\sin x$.

157. $\sqrt{1-x^2}+x\operatorname{arc\,sin} x$. **158.** $x\sin x+\cos x$. **159.** $-\sqrt{x}+(1+x)\arctan\sqrt{x}$.

160. (a) $\frac{x}{2}\sin 2x-\frac{2x^2-1}{4}\cos 2x$; (b) $-\frac{x}{2}+\frac{1+x^2}{2}\arctan x$; (c) $-\frac{1}{2}\times$

$\times e^{-2x}\left(x^2+x+\frac{1}{2}\right)$; (d) $-\frac{1}{2}(1+x^2)e^{-x^2}$; (e) $x(-1+\ln x)$; (f) $\frac{x^{n+1}}{n+1}\times$

$\times\left(\frac{-1}{n+1}+\ln x\right)$; (g) $\left(\frac{x^3}{3}+\frac{2x}{9}\right)\sinh 3x-\left(\frac{x^2}{3}+\frac{2}{27}\right)\cosh 3x$; (h) $x^2\cosh x-$

$-2x\sinh x+2\cosh x$; (i) $\ln\left|\tan\frac{x}{2}\right|-\cos x\ln\tan x$. **161.** (a) $\ln|x-2|+$

$+\ln|x+5|$; (b) $\frac{9}{343}\ln\left|\frac{x+5}{x-2}\right|-\frac{9}{49(x-2)}-\frac{5}{14(x-2)^2}$. **162.** (a) $\arctan(x-1)$; (b) $1/(1-x)$; (c) $\ln|x^2-2x+2|$. **163.** (a) $\frac{1}{2}\ln|x+1|-\frac{1}{4}\ln|x^2+1|+$ $+\frac{1}{2}\arctan x$; (b) $\frac{1}{50}\left[2\ln|x+2|-\ln(x^2+1)+14\arctan x+\frac{5(2x+1)}{x^2+1}\right]$. **164.** (a) $\frac{1}{2(1-x)}+\frac{1}{4}\ln\left|\frac{1+x}{1-x}\right|$; (b) $\frac{1}{2}\ln\left|\frac{(x+2)^4}{(x+1)(x+3)^3}\right|$; (c) $\frac{x^9}{9}-$ $-\frac{x^8}{8}+\frac{3x^7}{7}-\frac{5x^6}{6}+\frac{11x^5}{5}-\frac{21x^4}{4}+\frac{43x^3}{3}-\frac{85x^2}{2}+171x+\frac{1}{3}\ln\left|\frac{x-1}{(x+2)^{1024}}\right|$; (d) $\frac{1}{6}\ln\frac{(x+1)^2}{x^2-x+1}+\frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}$; (e) $\frac{1}{6}\ln\frac{(x-1)^2}{x^2+x+1}+$ $+\frac{1}{\sqrt{3}}\arctan\frac{2x+1}{3}$; (f) $\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\arctan x$; (g) $\frac{1}{4\sqrt{2}}\ln\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}+\frac{1}{2\sqrt{2}}\arctan\frac{x\sqrt{2}}{1-x^2}$. **165.** $2\sqrt{x}-2\ln(1+\sqrt{x})$. **166.** (a) $\frac{4}{1+\sqrt[4]{x}}+4\ln(1+\sqrt[4]{x})$; (b) $\frac{3}{4}\ln\frac{x\sqrt[3]{x}}{(1+\sqrt[6]{x})^2(1-\sqrt[6]{x}+2\sqrt[3]{x})^3}-$ $-\frac{3}{2\sqrt{7}}\arctan\frac{4\sqrt[6]{x}-1}{\sqrt{7}}$; (c) $\frac{2}{(1+\sqrt[4]{x})^2}-\frac{4}{1+\sqrt[4]{x}}$; (d) $\frac{x^2}{2}-\frac{x\sqrt{x^2-1}}{2}+$ $+\frac{1}{2}\ln|x+\sqrt{x^2-1}|$; (e) $\frac{1}{2}[x+2\sqrt{x}-\sqrt{x(1+x)}-\ln(\sqrt{x}+\sqrt{1+x})]$.

Hint. Multiply the numerator and denominator by the expression $\sqrt{1+x}-$ $-1-\sqrt{x}$; (f) $\frac{n}{b-a}\sqrt[n]{\frac{x-b}{x-a}}$.

167. $\ln|1+2x+2\sqrt{1+x+x^2}|$.

168. $\frac{x-1}{2}\sqrt{x^2-2x+2}+\frac{1}{2}\ln|x-1+\sqrt{x^2-2x+2}|$.

169. (a) $\frac{3x}{8}-\frac{\sin 2x}{4}+\frac{\sin 4x}{32}$; (b) $\frac{1}{2}\ln|\tan x|$.

170. (a) $\sin x-\frac{\sin^3 x}{3}$; (b) $\frac{1}{2}\ln\left|\tan\left(x+\frac{\pi}{4}\right)\right|$.

171. $\frac{1}{2}\ln\left|2\tan\frac{x}{2}-1\right|$. **172.** (a) $\frac{1}{ab}\arctan\left(\frac{a}{b}\tan x\right)$; (b) $\tan(x/2)$; (c) $-\cot(x/2)$; (d) $-\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)$; (e) $\frac{1}{2}\ln[\sinh(2x-1)\cosh(2x+1)]$ $\left(x>\frac{1}{2}\right)$. **174.** (a) Since $2x/\pi<\sin x<x$ $[0,\pi/2]$, we have

$$\int_0^{\pi/2}\frac{2x}{\pi}dx<\int_0^{\pi/2}\sin x\,dx<\int_0^{\pi/2}x\,dx;$$

(b) since $e^x \geq 1+x$ $[0, 1]$, we have $\int_0^1 e^x dx \geq \int_0^1 (1+x)\,dx$. **175.** (a) $\pi/6$; (b) 1; (c) 1; (d) $\ln\sqrt{2}$; (e) $\pi/3$; (f) 1; (g) 1; (h) $-\frac{3}{4}+\ln 4$.

176. $\pi/(2ab)$ (see Problem 172). **177.** π (see Problem 156). **178.** $\frac{1}{2}\ln\frac{e}{2}$ (see Problem 154). **179.** $f'(1)-f(1)+f(0)$. **181.** (a) $\sin x^2$; (b) $-\sqrt{1+a^2}$; (c) 0. **183.** 2. **184.** 4.5. **185.** $2bh/3$. **186.** πab **187.** $3\pi a^2/2$. **188.** $(e^2+1)/4$. **189.** $6a$. **190.** $8a$. **191.** Taking θ for the parameter, we have $x = \varrho\cos\theta = f(\theta)\cos\theta$, $y = \varrho\sin\theta = f(\theta)\sin\theta$,

$$|\Gamma| = \int_\alpha^\beta \sqrt{f^2(\theta)+(f'(\theta))^2}\,d\theta.$$

192. $3\pi a/2$. See Problem 191; one half of the curve is described when φ changes from 0 to $3\pi/2$. **193.** $8a$. The upper half of the cardioid is described when φ varies from 0 to π. **194.** (a) $\frac{1}{3}\pi r^2 h$; (b) $\frac{\pi h}{3}[r_1^2+r_2^2+r_1r_2]$; (c) $\pi^2/2$; (d) πpa^2. **195.** (a) $8\pi\left(\pi-\frac{4}{3}\right)a^2$; (b) 3π; (c) $\frac{\pi}{27}[10^{3/2}-1]$; (d) $\frac{12}{5}\pi a^2$. **196.** (a) By the method of rectangles the required integral

$$I \approx \frac{1}{8}\sum_{k=0}^{7}\frac{1}{1+\xi_k},$$

where

$$\xi_k = \frac{x_k+x_{k+1}}{2}, \quad x_k = \frac{k}{8} \qquad (k = 0, 1, \ldots, 8).$$

Substituting the values of ξ_k, we obtain

$$I \approx 2\sum_{k=8}^{15}\frac{1}{2k+1} \approx 0.6927.$$

The remainder

$$R_8 \leq \frac{1}{8^2}\cdot\frac{2}{12} = \frac{1}{384} \leq 0.0027.$$

By the trapezoid method

$$I \approx \frac{1}{16}\{f(x_0)+2f(x_1)+\ldots+2f(x_7)+f(x_8)\} \approx 0.6941$$

with the same error as for the method of rectangles.

By Simpson's method,

$$I \approx \frac{1}{24}\{f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\ldots+4f(x_7)+f(x_8)\} \approx 0.69315$$

with the remainder of the quadrature formula

$$R_4 \leq \frac{M_4}{4^4\cdot 2880} \leq 5\cdot 10^{-5} \leq 10^{-4}.$$

Thus, Simpson's rule gives the first three correct digits for the value of the integral. The exact value of the integral

$$I = \int_0^1 \frac{dx}{1+x} = \ln 2 = 0.693147 \ldots .$$

(b) By the method of rectangles, $I \approx 0.8358$ and by the trapezoid rule, $I \approx 0.8352$ with the remainder $R_{12} \leqslant 0.004$. By Simpson's rule, the remainder $R_6 \leqslant \frac{10}{2880 \cdot 6^4} \leqslant 10^{-5}$. Therefore, evaluating I by Simpson's formula, we obtain four correct digits: $I \approx 0.83565$.

197. $L_3(x) = -\frac{1}{2}(x-3)x^2$. **198.** (a) $\pi/4$; (b) $\frac{1}{1-\alpha}$, $0 < \alpha < 1$; ∞, $\alpha \geqslant 1$; (c) $\pi/2$; (d) ∞. **199.** All integrals converge. **200.** (a) 1; (b) $\frac{a}{a^2+b^2}$; (c) $\frac{1}{2}(e^{\pi}+1)$. **201.** (a) Converges for $p > 0$, $a \neq 0$; for $a = 0$ it is convergent if $p > 1$; (b) convergent if either p or q exceeds zero; (c) convergent for $m > -1$, $n-m > 1$.

Chapter 3.

202. 2. **203.** -2. **204.** 1. **205.** 1. **206.** $\cos(\alpha+\beta)$. **207.** $4ab$. **208.** 1. **209.** 4. **210.** 0. **211.** (a) Odd; (b) even; (c) even; (d) odd. **212.** $A_{11} = bc-x^2$, $A_{12} = x^2-cx$, $A_{13} = x^2-bx$, $A_{21} = x^2-cx$, $A_{22} = ac-x^2$, $A_{23} = x^2-ax$, $A_{31} = x^2-bx$, $A_{32} = x^2-ax$, $A_{33} = ab-x^2$; $\Delta = a_{11}A_{11}+a_{12}A_{12}+a_{13}A_{13} = 2x^3-x^2(a+b+c)+abc$.

213. (a) 0; (b) -4. **214.** (a) 72; (b) $(c-a)(b-a)(c-b)$.

215. $\Delta = \Delta_1 \cdot \Delta_2 =$

rows by columns:

$$\begin{vmatrix} 7 & 10 & 9 \\ 11 & 14 & 10 \\ 9 & 13 & 9 \end{vmatrix} = 35;$$

rows by rows:

$$\begin{vmatrix} 7 & 10 & 9 \\ 11 & 14 & 10 \\ 9 & 13 & 9 \end{vmatrix} = 35;$$

columns by rows:

$$\begin{vmatrix} 7 & 10 & 10 \\ 12 & 14 & 11 \\ 8 & 11 & 9 \end{vmatrix} = 35;$$

columns by columns:

$$\begin{vmatrix} 7 & 10 & 10 \\ 12 & 14 & 11 \\ 8 & 11 & 9 \end{vmatrix} = 35.$$

217. rank $A = 2$, $A^* = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 2 \\ 1 & 1 & 1 \end{pmatrix}$.

218. $x_1 = -7, x_2 = 5$. **219.** $x_1 = 3/2, x_2 = -1/2$. **220.** $x = 3/2, y = -1/2$. **221.** The system has no solution. **222.** $x = -84, y = -93/2, z = 31/2$. **223.** The system has an infinitude of solutions:

$$B = \begin{pmatrix} 2 & 7 & 3 & 1 & 5 \\ 1 & 3 & 5 & -2 & 3 \\ 1 & 5 & -9 & 8 & 1 \\ 5 & 18 & 4 & 5 & 12 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & -7 & 5 & -1 \\ 1 & 3 & 5 & -2 & 3 \\ 0 & 2 & -14 & 10 & -2 \\ 0 & 3 & -21 & 15 & -3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 1 & -7 & 5 & -1 \\ 1 & 3 & 5 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & -7 & 5 & -1 \\ 1 & 3 & 5 & -2 & 3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 1 & -7 & 5 & -1 \\ 1 & 0 & 26 & -17 & 6 \end{pmatrix}.$$

Hence,

$$x = 6 - 26z + 17t, \quad y = -1 + 7z - 5t,$$

where z and t are arbitrary numbers.

224. rank $A = 3$, rank $B = 3$. **225.** (a) $5/\sqrt{2}$; (b) $x = \sqrt{2}, y = 1, z = -1$; (c) $1/2$. **226.** $3, \sqrt{6}; \sqrt{11}$; 2. **227.** (a) $5/21$; (b) $1/\sqrt{2}$; (c) $1/\sqrt{2}$.

228. It cannot, since in this case $\cos^2\alpha + \cos^2\beta + \cos^2\gamma \geqslant \cos^2\alpha + \cos^2\beta = 5/4 > 1$, which is impossible. **229.** $x = y = z = \sqrt{3}$. **230.** $|a - b| = 22$. *Hint.* $|\mathbf{a} \pm \mathbf{b}| = \sqrt{|\mathbf{a}|^2 + |\mathbf{b}|^2 \pm 2(\mathbf{a}, \mathbf{b})}$. **231.** $|\mathbf{a} + \mathbf{b}| = 20$. **232.** $\omega = \pi/4$. **233.** $\lambda = -\dfrac{(\mathbf{a}, \mathbf{a})}{(\mathbf{b}, \mathbf{b})} = -\left(\dfrac{|\mathbf{a}|}{|\mathbf{b}|}\right)^2$. **234.** $x = 2/5, y = z = 4/5$. **235.** $(1, -2)$. *Hint.* This problem is equivalent to dividing the line segment AB into two equal parts. **236.** $x = 1/2, y = -5/4$. **237.** Let $M_1 = (z_1, z_2)$ and $M_2 = (\bar{z}_1, \bar{z}_2)$ be the division points (Fig. 62). We determine the numbers μ and λ (see [2], Sec. 7) for the point M_1:

$$\mu = \frac{|M_1A|}{|AB|} = \frac{1}{3}, \quad \lambda = \frac{|M_1B|}{|AB|} = \frac{2}{3}.$$

Therefore

$$z_1 = 1 \cdot \lambda + 4\mu = \frac{2}{3} + \frac{4}{3} = 2,$$

$$z_2 = -5\lambda + 3\mu = -\frac{10}{3} + 1 = -\frac{7}{3},$$

$$M_1 = \left(2, -\frac{7}{3}\right).$$

Analogously,

$$\bar{z}_1 = 3, \quad \bar{z}_2 = 1/3 \quad (\mu = 2/3, \lambda = 1/3),$$

$$M_2 = (3, 1/3).$$

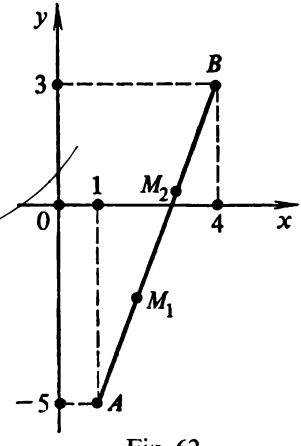

Fig. 62.

238. Of the three given points, only M_2 lies on the indicated line. **239.** $y = -\frac{2}{3}x+\frac{13}{3}$. The point $M_2 = (2, 3)$ lies on the given line, therefore $y-3 = -\frac{2}{3}(x-2)$. If for the point lying on the given line we take another point, for instance $M_0 = (0, 13/3)$, then the equation takes the form: $y-\frac{13}{3} = -\frac{2}{3}x$. **240.** (a) $-x-2y+4 = 0$; (b) $-2x+y+3 = 0$. **241.** $2x-3y = 0$, $3x+2y = 0$. **242.** (a) $\frac{-2x}{\sqrt{29}}-\frac{5y}{\sqrt{29}}-\frac{4}{\sqrt{29}} = 0$; (b) $\frac{x}{\sqrt{2}}+\frac{y}{\sqrt{2}}+\frac{1}{\sqrt{2}} = 0$; (c) $-\frac{x}{\sqrt{5}}+\frac{y}{\sqrt{5}}-\frac{3}{\sqrt{5}} = 0$. **243.** (a) $d = \frac{|2\cdot1+4\cdot2-5|}{\sqrt{2^2+4^2}} = \frac{\sqrt{5}}{2}$; (b) $d = \frac{|-2\cdot1-8\cdot2-1|}{\sqrt{68}} = \frac{19}{\sqrt{68}}$; (c) $d = \frac{|1\cdot1+1\cdot2|}{\sqrt{2}} = \frac{3}{\sqrt{2}}$. **244.** $(x-1)-2(y-2)+3(z+3) = 0$. **245.** $-4x-y+2z+3 = 0$. **246.** (a) $A(x-1)+B(y-1)-2(A+2B)(z-1) = 0$, where A, B are arbitrary numbers, not all zero; (b) $2(x-1)+4(y-1)+4(z-1) = 0$. **247.** Cases (a), (b), and (c) define parallel planes. In Case (c) we have even two merged planes. In Case (d) the planes are not parallel. **248.** (a) $\frac{2}{7}x-\frac{3}{7}y-\frac{6}{7}z-1 = 0$; (b) $\frac{4}{9}x-\frac{1}{9}y+\frac{8}{9}z-\frac{14}{9} = 0$. **249.** (a) $d = \frac{1}{7}\times$ $\times|2\cdot1-3\cdot2+6\cdot1-7| = \frac{5}{7}$; (b) $d = \frac{1}{3}|2\cdot1+1\cdot2-2\cdot1-1| = \frac{1}{3}$. **250.** $\cos\varphi = 59/63$. **251.** $x^2+y^2+z^2 = 144/29$. *Hint.* The radius of the sphere is the distance from the origin to the given plane ($d = 12/\sqrt{29}$). **252.** $\frac{x}{3}+\frac{y}{6}+\frac{z}{-6/5} = 1$. **253.** $-7(x-2)+(y+1)+5(z-1) = 0$. *Hint.* From the conditions of the orthogonality of the planes find the ratios A/C and B/C, where A, B, C are the coefficients of the required plane. **254.** (a) $\alpha = \pi/3$, $\beta = \pi/4$, $\gamma = \pi/3$; (b) $\alpha = \pi/6$, $\beta = \pi/3$, $\gamma = 0$. **255.** (a) 5/3; (b) 3/14. *Hint.* Take a point on one of the planes and find its distance to the other. **256.** (2, −1, 0); (4/3, 0, −1/3); (0, 2, −1). **257.** (a) $A_1D_2 = A_2D_1$; (b) $A_1 = D_1 = 0$, $A_2 = D_2 = 0$. **258.** $\frac{x-1}{2} = \frac{y-0}{-3} = \frac{z+1}{5}$. **259.** $x = 1+2t$, $y = -1+t$, $z = -3+5t$. **260.** (a) $\frac{x-2}{2} = \frac{y+1}{7} = \frac{z}{4}$; (b) $\frac{x}{-5} = \frac{y+1}{12} = \frac{z-1}{13}$. *Hint.* In Case (a) solve the system with respect to x and y, and in Case (b) with respect to z and y. **261.** $\cos\varphi = \frac{1\cdot1-1\cdot1+\sqrt{2}\sqrt{2}}{2\cdot2} = \frac{1}{2}$, $\varphi = \frac{\pi}{3}$. **262.** $l = 3$. *Hint.* Pass to parametric representation of the straight lines. Supposing that the lines intersect at some point, we obtain the system

$$\begin{cases} 2t_0-lt_1 = 5, \\ 3t_0+4t_1 = -1, \\ 4t_0-2t_1 = 6, \end{cases}$$

from which we find l, t_0, t_1. **263.** $\frac{x-1}{2} = \frac{y+1}{-4} = \frac{z-0}{1}$. *Hint.* The vector (2, −4, 1) is collinear with the vector $\mathbf{a} = (a_1, a_2, a_3)$ lying on the straight line. **264.** (a) The vectors $\mathbf{a}$ and $\mathbf{b}$ are oriented in an opposite way as compared

with the coordinate system; (b) the vectors **a** and **b** are oriented in the same way as the coordinate system (the determinant formed from the coordinates of the vectors $\begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix} = 1 > 0$). **265.** $|\mathbf{a}\times\mathbf{b}| = 21$. **266.** Yes ($(\mathbf{a}\times\mathbf{b}) = \mathbf{0}$). **267.** The vectors **a** and **b** must be collinear. **269.** $\sin\varphi = \frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}|\cdot|\mathbf{b}|} = \frac{5\sqrt{17}}{21}$. **270.** $S = \left\| \begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix} \right\| = 2$. **271.** (a) Yes (coplanar); (b) no. **273.** (a) Linearly dependent (see Problem 224, matrix B); (b) linearly independent, the rank of the matrix made up from the coordinate of the vectors is equal to three. **275.** $\lambda = 15$.

276. (a) $AB = \begin{pmatrix} 5 & 2 \\ 7 & 0 \end{pmatrix}$, $BA = \begin{pmatrix} 29 & -22 \\ 31 & -24 \end{pmatrix}$; (b) $AB = \begin{pmatrix} 1 & 3 \\ 3 & 7 \end{pmatrix}$, $BA = \begin{pmatrix} 4 & 6 \\ 3 & 4 \end{pmatrix}$;

(c) $AB = \begin{pmatrix} 43 & 10 \\ 49 & 11 \\ 37 & 9 \end{pmatrix}$; the product BA has no sense.

277. (a) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$; (b) $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$; (c) $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ $(n > 3)$.

278. (a) $B = \frac{1}{2}\begin{pmatrix} 2a & 2b \\ 3b & 3b+2a \end{pmatrix}$; (b) $B = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$.

279. (a) $A^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$; (b) $A^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ -38 & 41 & -34 \\ 27 & -29 & 24 \end{pmatrix}$;

(c) $A^{-1} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} = A^*$. **281.** $X = \begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix}$.

282. (1) $A^{-1} = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$; (2) $A^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

283. (a) Operator A is a linear one. Its matrix has the form $A = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \\ 3 & -1 & 1 \end{pmatrix}$. *Hint.* The column of A represent the coordinates of the images of the basis vectors. For example, $\mathbf{e}^1 = (1, 0, 0)$. $A\mathbf{e}^{-1} = (0, 2, 3)$; (b) operator A is not a linear one:

$$A(\mathbf{x}+\mathbf{y}) = (x_1+y_1,\ x_2+y_2+1,\ x_3+y_3+1) \neq A\mathbf{x}+A\mathbf{y} = \\ = (x_1+y_1,\ x_2+y_2+2,\ x_3+y_3+2).$$

284. $BA^{-1} = \begin{pmatrix} 2 & -11 & 6 \\ 1 & -7 & 4 \\ 2 & -1 & 0 \end{pmatrix}$. **285.** $BA^{-1} = \frac{1}{3}\begin{pmatrix} -6 & 11 & 5 \\ -12 & 13 & 10 \\ 6 & -5 & -5 \end{pmatrix}$.

286. (a) $\begin{pmatrix} 1 & 0 & 2 & 1 \\ 2 & 3 & 5 & 1 \\ 3 & -1 & 0 & 2 \\ 1 & 1 & 2 & 3 \end{pmatrix}$.

Hint. In the old basis $A\mathbf{i}^1 = \mathbf{i}^1 + 3\mathbf{i}^2 + 2\mathbf{i}^3 + \mathbf{i}^4$. This can be written as $A\mathbf{i}^1 = \mathbf{i}^1 + 2\mathbf{i}^3 + \mathbf{i}^4$, therefore the first column of the new matrix will consist of the following elements: 1, 2, 3, 1. Then we also transform $A\mathbf{i}^3$, $A\mathbf{i}^2$, $A\mathbf{i}^4$.

(b) $\begin{pmatrix} -2 & 0 & 1 & 0 \\ 1 & -4 & -8 & -7 \\ 1 & 4 & 6 & 4 \\ 1 & 3 & 4 & 7 \end{pmatrix}$.

Hint. The transformation of $A\mathbf{i}^1$ to the necessary form leads us to the solution of the system:

$$A\mathbf{i}^1 = \mathbf{i}^1 + 3\mathbf{i}^2 + 2\mathbf{i}^3 + \mathbf{i}^4 =$$
$$= \alpha\mathbf{i}^1 + \beta(\mathbf{i}^1 + \mathbf{i}^2) + \gamma(\mathbf{i}^1 + \mathbf{i}^2 + \mathbf{i}^3) + \delta(\mathbf{i}^1 + \mathbf{i}^2 + \mathbf{i}^3 + \mathbf{i}^4).$$

Then $A(\mathbf{i}^1 + \mathbf{i}^2)$, $A(\mathbf{i}^1 + \mathbf{i}^2 + \mathbf{i}^3)$, $A(\mathbf{i}^1 + \mathbf{i}^2 + \mathbf{i}^3 + \mathbf{i}^4)$ are transformed in a similar way.

287. $\begin{pmatrix} -5 & -10 \\ 1 & 3 \end{pmatrix}$. *Hint.* We have $\mathbf{a}^1 = \mathbf{i}^1 + 2\mathbf{i}^2$, $\mathbf{a}^2 = -\mathbf{i}^1 + \mathbf{i}^2$, $\mathbf{b}^1 = \mathbf{i}^1 - 2\mathbf{i}^2$, $\mathbf{b}^2 = 3\mathbf{i}^1 - \mathbf{i}^2$, where $(\mathbf{i}^1, \mathbf{i}^2)$ is the original basis of the space. Express $\mathbf{a}^1$, and $\mathbf{a}^2$ in terms of $\mathbf{b}^1$, $\mathbf{b}^2$: $\mathbf{a}^1 = -\frac{7}{5}\mathbf{b}^1 + \frac{4}{5}\mathbf{b}^2$, $\mathbf{a}^2 = -\frac{2}{5}\mathbf{b}^1 - \frac{1}{5}\mathbf{b}^2$. Then find $A\mathbf{b}^1$, $A\mathbf{b}^2$ expressed in terms of $\mathbf{b}^1$, $\mathbf{b}^2$.

This problem can also be solved in the matrix form. Let $a = (\mathbf{a}^1, \mathbf{a}^2)$, $b = (\mathbf{b}^1, \mathbf{b}^2)$. T is the transformation matrix from basis b to basis a, the columns of T consisting of the coordinates of the vectors $\mathbf{a}^1$, $\mathbf{a}^2$ in the basis $\mathbf{b}^1$, $\mathbf{b}^2$. Then we can write in the matrix form

$$a = bT, \tag{1}$$

where in the present case

$$T = \begin{pmatrix} -\frac{7}{5} & -\frac{2}{5} \\ \frac{4}{5} & -\frac{1}{5} \end{pmatrix}.$$

Now let the linear transformation φ in basis a be given by matrix A and in basis b by matrix B:

$$\varphi(a) = aA, \quad \varphi(b) = bB, \tag{2}$$

where the columns of matrices contain the coordinates of the images of the basis vectors in the corresponding basis, $\varphi(a) = (\varphi(\mathbf{a}^1), \varphi(\alpha^2))$, $\varphi(b) = (\varphi(\mathbf{b}^1), \varphi(\mathbf{b}^2))$. It is obvious that

$$\varphi(a) = \varphi(b)\,T. \tag{3}$$

Therefore, from (2) and (1) we have

$$\varphi(b)\,T = bBT \quad \text{and} \quad \varphi(a) = bTA,$$

whence, by virtue of (3), we have $bBT = bTA$.

Thus, $BT = TA$,

$$B = TAT^{-1}. \tag{4}$$

Let us now find matrix B by formula (4). It is easy to compute that

$$T^{-1} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ -\frac{4}{5} & -\frac{7}{3} \end{pmatrix}, \quad AT^{-1} = \begin{pmatrix} \frac{7}{3} & \frac{16}{3} \\ \frac{13}{3} & \frac{19}{3} \end{pmatrix}, \quad B = TAT^{-1} = \begin{pmatrix} -5 & -10 \\ 1 & 3 \end{pmatrix}.$$

288. $B = TAT^{-1} = \begin{pmatrix} \frac{71}{15} & \frac{98}{15} \\ -\frac{32}{15} & -\frac{41}{15} \end{pmatrix}$.

289. (a) Orthogonal, (b) and (c) not orthogonal.

290. The vectors $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$ form an orthogonal basis in R_3, since the rank of the matrix constructed from the coordinates of the vectors is equal to three (i.e. $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$ are linearly independent) and the vectors are pairwise orthogonal; $\mathbf{x} = \frac{1}{14}\mathbf{e}^1 - \frac{1}{14}\mathbf{e}^3$.

291. $\mathbf{z} = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$, $\mathbf{t} = \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$.

292. (a) The basis is oriented in an opposite way: (b) the basis is oriented in the same way as $\mathbf{i}^1$, $\mathbf{i}^2$, $\mathbf{i}^3$ ($\Delta = 1$).

293. $x_1 = \frac{\sqrt{3}}{2}x_1' - \frac{1}{2}x_2'$, $x_2 = \frac{1}{2}x_1' + \frac{\sqrt{3}}{2}x_2'$, i.e. the passage (x_1', x_2') to (x_1, x_2) is realized with the aid of the rows of matrix A^*.

294. The passage from the coordinates (x_1, x_2) to the coordinates (x_1', x_2') in the new basis is realized with the aid of the rows of the matrix $(A^*)^{-1}$, and the passage from the coordinates (x_1', x_2') to (x_1, x_2) with the aid of the rows of matrix A^*. We have

$$A^* = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \quad (A^*)^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix},$$

therefore

$$\begin{cases} x_1 = x_1' + 2x_2', \\ x_2 = x_1' + x_2', \end{cases} \quad \begin{cases} x_1' = -x_1 + 2x_2, \\ x_2' = x_1 - x_2. \end{cases}$$

295. (a) No; (b) no; (c) yes; (d) no; (e) no; (f) no, if the given line does not pass through the origin; (g) yes; (h) yes.

296. The entire plane; the vectors lying on any straight line passing through the origin; the origin.

297. The set of vectors lying on the straight line $x_2 = -\frac{x_1}{k}$ ($k \neq 0$); $x_1 = 0$ for $k = 0$.

298. (a) The dimension is equal to 3 (the rank of the matrix formed from the coordinates of the vectors is equal to 3). A basis is formed, for instance, by the vectors $\mathbf{a}^1$, $\mathbf{a}^2$, $\mathbf{a}^4$; (b) the dimension is equal to 2. A basis is formed by any two vectors of the system.

299. Subspace L' consists of the vectors $\mathbf{v} = (x_1, x_2, x_3, x_4)$ for which $(\mathbf{v}, \mathbf{a}^1) = (\mathbf{v}, \mathbf{a}^2) = 0$, i.e. the coordinates of the vectors $\mathbf{v}$ satisfy the condition $x_4 = x_1$, $2x_1+x_2+x_3 = 0$. The vector $\mathbf{a} = (y_1, y_2, y_3, y_4)$ is orthogonal to all the vectors $\mathbf{v} \in L'$ therefore its coordinates satisfy the condition

$$(y_1+y_4-2y_3)x_1+(y_2-y_3)x_2 = 0, \quad \forall x_1, x_2.$$

Hence, $y_2 = y_3$, $y_1+y_4-2y_3 = 0$. For the numbers α and β we obtain the system $\alpha+2\beta = y_1$, $\beta = y_2$, $\beta = y_3$, $-\alpha = y_4$. This system is solvable for the indicated y_1, y_2, y_3, y_4, namely $\alpha = -y_4$, $\beta = y_2 = y_3$ (the equality $\alpha+2\beta = y_1$ is fulfilled automatically).

301. $\begin{pmatrix} 3 & 6 \\ 0 & 1 \end{pmatrix}$. **302.** $A^*(f) = \begin{pmatrix} -36 & -37 & -15 \\ 30 & 30 & 14 \\ 26 & 27 & 9 \end{pmatrix}$.

303. (a) $\lambda_1 = 2$. *Hint.* Investigate for extremum the quadratic form $u = x^2+y^2+2xy$ in the unit circle $x^2+y^2 = 1$; (b) $\lambda_1 = 3$.

304. (a) The form is indeterminate by sign, since $\Delta_2 = -3 < 0$; (b) the form is indeterminate by sign ($\Delta_2 = -1 < 0$); (c) the form is strictly positive ($\Delta_1 = 2 > 0$, $\Delta_2 = 1 > 0$, $\Delta_3 = 2 > 0$).

305. (a) $\lambda_1 = \frac{1}{2}\left[a_{11}+a_{22}+\sqrt{4a_{12}^2+(a_{11}-a_{22})^2}\right]$

$$= \frac{1}{2}\left[1-1+\sqrt{4\cdot 4+4}\right] = \sqrt{5},$$

$$\lambda_2 = \frac{1}{2}\left[a_{11}+a_{22}-\sqrt{4a_{12}^2+(a_{11}-a_{22})^2}\right] = -\sqrt{5},$$

the form is of hyperbolic type; (b) $\lambda_1 = \frac{27+\sqrt{725}}{2} > 0$, $\lambda_2 = \frac{27-\sqrt{725}}{2} > 0$; the form is of elliptic type; (c) $\lambda_1 = 4 > 0$, $\lambda_2 = 0$; the form is of the parabolic type.

306. (a) The characteristic equation has the form

$$\begin{vmatrix} -\lambda & 2 & 2 \\ 2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0.$$

The roots of the equation: $\lambda_1 = \lambda_2 = 4$, $\lambda_3 = -2$. The canonical form of the form: $4\xi_1^2+4\xi_2^2-2\xi_3^2$; (b) $8\xi_1^2+8\xi_2^2+5\xi_3^2$ is the canonical form of the form.

307. (a) The characteristic equation is

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 5-\lambda & 0 \\ 2 & 0 & 7-\lambda \end{vmatrix} = (6-\lambda)(5-\lambda)(7-\lambda)-4(5-\lambda)-4(7-\lambda) = 0,$$

or

$$(6-\lambda)(5-\lambda)(7-\lambda)-8(6-\lambda) = 0,$$

has the roots $\lambda_1 = 9$, $\lambda_2 = 6$, $\lambda_3 = 3$. The eigenvector $\mathbf{x}^1$ is found from the system

$$\left.\begin{aligned} -3x_1-2x_2+2x_3 &= 0, \\ -2x_1-4x_2 &= 0, \\ 3x_1 \qquad -2x_3 &= 0. \end{aligned}\right\}$$

Hence, $x_3 = x_1$, $-2x_2 = x_1$. The vector $\mathbf{y}^1 = \left(x_1, -\frac{x_1}{2}, x_1\right)$ is a solution of the system. Normalizing this vector, we obtain

$$\mathbf{x}^1 = \frac{\mathbf{y}^1}{|\mathbf{y}^1|} = \left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right).$$

Analogously, we obtain

$$\mathbf{x}^2 = \left(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right), \quad \mathbf{x}^3 = \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right).$$

The canonical form of the form: $9\xi_1^2 + 6\xi_2^2 + 3\xi_3^2$. Orthogonal transformation:

$$x_1 = \frac{2}{3}\xi_1 - \frac{1}{3}\xi_2 + \frac{2}{3}\xi_3, \quad x_2 = -\frac{1}{3}\xi_1 + \frac{2}{3}\xi_2 + \frac{2}{3}\xi_3,$$

$$x_3 = \frac{2}{3}\xi_1 + \frac{2}{3}\xi_2 - \frac{1}{3}\xi_3.$$

(b) $18\xi_1^2 + 18\xi_2^2 + 9\xi_3^2$; $x_1 = \frac{2}{3}\xi_1 - \frac{2}{3}\xi_2 + \frac{1}{3}\xi_3$,

$$x_2 = -\frac{2}{3}\xi_1 - \frac{1}{3}\xi_2 + \frac{2}{3}\xi_3, \quad x_3 = \frac{1}{3}\xi_1 + \frac{2}{3}\xi_2 + \frac{2}{3}\xi_3.$$

308. (a) $AC - B^2 = 9 > 0$—a curve of the elliptic type; $3(x-1)^2 + 3(y-2)^2 = 12$; $\xi = x-1$, $\eta = y-2$; $3\xi^2 + 3\eta^2 = 12$—a circle of radius 2.

(b) $AC - B^2 = 6 > 0$—a curve of the elliptic type; $3\xi^2 + 2\eta^2 = 6$—an ellipse with the semiaxes $a = \sqrt{2}$, $b = \sqrt{3}$.

(c) $AC - B^2 = -2 < 0$—a curve of the hyperbolic type; $\xi^2 - 2\eta^2 = 2$—a hyperbola with the semiaxes $a = \sqrt{2}$, $b = 1$.

(d) $AC - B^2 = 9 > 0$—a curve of the elliptic type; $3\xi^2 + 3\eta^2 = 0$—the point (0, 0).

(e) $AC - B^2 = -2 < 0$—a curve of the hyperbolic type; $\xi^2 - 2\eta^2 = 0$—a pair of intersecting lines $\xi - \sqrt{2}\eta = 0$, $\xi + \sqrt{2}\eta = 0$.

(f) $AC - B^2 = 0$—a curve of the parabolic type; $4\xi - 3\eta^2 = 0$—a parabola with the axis of symmetry ξ.

(g) $AC - B^2 = 6 > 0$—a curve of the elliptic type; $3\xi^2 + 2\eta^2 = -1$—an imaginary ellipse.

309. (a) $AC - B^2 = -16 < 0$—a curve of the hyperbolic type; $\lambda_1 = 8$, $\lambda_2 = -2$;

$$\mathbf{x}^1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad \mathbf{x}^2 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right); \quad \begin{cases} x = \frac{1}{\sqrt{2}}(\xi - \eta), \\ y = \frac{1}{\sqrt{2}}(\xi + \eta); \end{cases}$$

$$8\xi^2 - 2\eta^2 - \frac{2}{\sqrt{2}}(\xi - \eta) - \frac{14}{\sqrt{2}}(\xi + \eta) - 13 = 0$$

is the equation of the curve in the system (ξ, η). This equation can be written as follows:

$$8\left(\xi - \frac{1}{\sqrt{2}}\right)^2 - 2\left(\eta + \frac{3}{\sqrt{2}}\right)^2 = 8.$$

The translation

$$u = \xi - \frac{1}{\sqrt{2}}, \quad v = \eta + \frac{3}{\sqrt{2}}$$

reduces the equation to the form

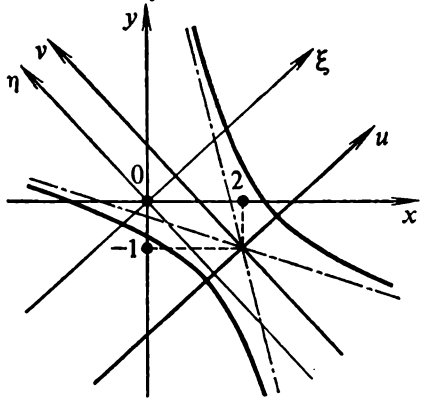

Fig. 63.

$$u^2 - \frac{v^2}{4} = 1.$$

This is a hyperbola (Fig. 63) with the transverse (real) axis u. The general transformation of the coordinates has the form

$$x = \frac{1}{\sqrt{2}}\left(u - v + \frac{4}{\sqrt{2}}\right),$$

$$y = \frac{1}{\sqrt{2}}\left(u + v - \frac{2}{\sqrt{2}}\right).$$

(b) $AC - B^2 = 576 > 0$—a curve of the elliptic type;

$\lambda_1 = 32, \quad \lambda_2 = 18, \quad B < 0;$

$$\mathbf{x}^1 = \left(\frac{1}{\sqrt{2}}, \ -\frac{1}{\sqrt{2}}\right), \quad \mathbf{x}^2 = \left(-\frac{1}{\sqrt{2}}, \ -\frac{1}{\sqrt{2}}\right);$$

$$\begin{cases} x = \dfrac{1}{\sqrt{2}}(\xi - \eta), \\ y = -\dfrac{1}{\sqrt{2}}(\xi + \eta); \end{cases}$$

$$32\xi^2 + 18\eta^2 + \frac{64}{\sqrt{2}}(\xi - \eta) + \frac{64}{\sqrt{2}}(\xi + \eta) - 224 = 0,$$

$$32(\xi + \sqrt{2})^2 + 18\eta^2 = 288;$$

$$\frac{(\xi + \sqrt{2})^2}{9} + \frac{\eta^2}{16} = 1; \quad u = \xi + \sqrt{2}, \quad v = \eta;$$

$\frac{u^2}{9} + \frac{v^2}{16} = 1$—an ellipse with the semiaxes $a = 3$, $b = 4$ (Fig. 64);

$$x = \frac{1}{\sqrt{2}}(u - v - \sqrt{2}), \quad y = -\frac{1}{\sqrt{2}}(u + v - \sqrt{2}).$$

(c) $AC - B^2 = 0$—a curve of the parabolic type; $\lambda_1 = 25$, $\lambda_2 = 0$, $B < 0$;

$$\mathbf{x}^1 = \left(\frac{3}{5}, \ -\frac{4}{5}\right), \quad \mathbf{x}^2 = \left(-\frac{4}{5}, \ -\frac{3}{5}\right);$$

$$\begin{cases} x = \dfrac{1}{5}(3\xi - 4\eta), \\ y = -\dfrac{1}{5}(4\xi + 3\eta); \end{cases}$$

$$25\xi^2 - \frac{20}{5}(3\xi - 4\eta) - \frac{110}{5}(4\xi + 3\eta) - 50 = 0, \quad (\xi - 2)^2 = 2(\eta + 3); \quad u = \xi - 2,$$

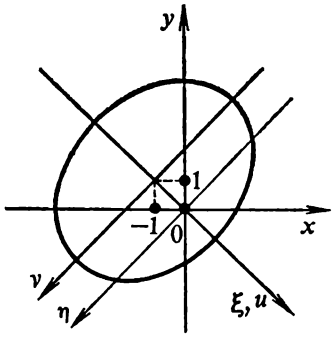

Fig. 64.

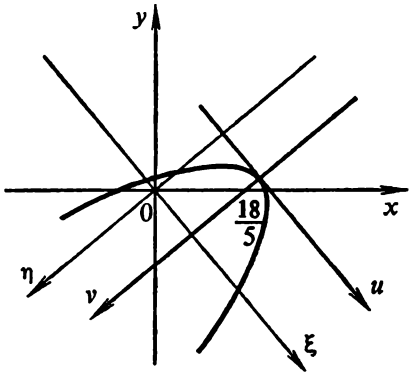

Fig. 65.

$v = \eta + 3$; $u^2 = 2v$—a parabola with the axis of symmetry v (Fig. 65); $x = \frac{1}{5}(3u - 4v + 18)$, $y = -\frac{1}{5}(4u + 3v - 1)$.

310. $AC - B^2 = 0$—a curve of the parabolic type; solving simultaneously the equations of the straight line and the curve, we obtain the equation

$$x^2(2-k)^2 + 6x + 1 = 0.$$

The discriminant of this equation has the form

$$9 - (2-k)^2 = (1+k)(5-k) \quad (k \neq 2).$$

(a) Therefore for $k = -1$ and $k = 5$ the straight line has one common point with the curve. For $k = 2$ the straight line $y = 2x$ and our curve also have one common point; (b) $-1 < k < 5$, $k \neq 2$; (c) $k < -1$, $k > 5$.

311. $k = -3$, $k = -1/3$.

312. $x^2 + 2xy + y^2 - x - 3y = 0$ (parabola, $AC - B^2 = 0$).

313. (a) $(x+1)^2 + (y+2)^2 + z^2 = 9$; $x + 1 = \xi$, $y + 2 = \eta$, $z = \zeta$; $\xi^2 + \eta^2 + \zeta^2 = 9$—the surface of a ball of radius 3.

(b) $\frac{\xi^2}{4} + \frac{\eta^2}{2} + \frac{\zeta^2}{4} = 1$—an ellipsoid with the semiaxes, $a = 2$, $b = \sqrt{2}$, $c = 2$.

(c) $\frac{\xi^2}{4} + \frac{\eta^2}{2} - \frac{\zeta^2}{4} = 1$—a hyperboloid of one sheet with the semiaxes $a = 2$, $b = \sqrt{2}$, $c = 2$.

(d) $\xi^2 + 2\eta^2 = 2\zeta$—an elliptic paraboloid ($p = 1$, $q = 1/2$).

(e) $\frac{\xi^2}{4} - \eta^2 - \frac{\zeta^2}{4} = 1$—a hyperboloid of two sheets with the semiaxes $a = 2$, $b = 1$, $c = 2$.

(f) $\frac{\xi^2}{4} + \frac{\eta^2}{2} = 1$—an elliptic cylinder (the equation contains no ζ).

314. (a) The characteristic equation has the form

$$\begin{vmatrix} 11-\lambda & 8 & 2 \\ 8 & 5-\lambda & -10 \\ 2 & -10 & 2-\lambda \end{vmatrix} = -\lambda^3 + 18\lambda^2 + 81\lambda - 1458$$

$$= \lambda^2(18-\lambda) + 81(\lambda - 18) = (\lambda^2 - 81)(18 - \lambda) = 0.$$

Its roots: $\lambda_1 = 18$, $\lambda_2 = 9$, $\lambda_3 = -9$. Find the eigenvector from the system

$$\begin{cases} (11-\lambda_1)x_1 + 8x_2 + 2x_3 = 0, \\ 8x_1 + (5-\lambda_1)x_2 - 10x_3 = 0, \\ 2x_1 - 10x_2 + (2-\lambda_1)x_3 = 0, \end{cases}$$

$$\begin{cases} -7x_1 + 8x_2 + 2x_3 = 0, \\ 8x_1 - 13x_2 - 10x_3 = 0, \\ 2x_1 - 10x_2 - 16x_3 = 0. \end{cases}$$

The rank of the matrix formed from the coefficients of the system is equal to two (all the three eigenvalues are different). Therefore we solve the system of two equations (in this case of any two):

$$\begin{cases} -7x_1 + 8x_2 = -2x_3, \\ 8x_1 - 13x_2 = 10x_3, \end{cases} \quad x_1 = -2x_3, \quad x_2 = -2x_3.$$

The vector $\mathbf{v} = (-2x_3, -2x_3, x_3)$ is a solution of the system; normalizing it, we get the eigenvector

$$\mathbf{x}^1 = (2/3, 2/3, -1/3).$$

We find in a similar way:

$$\mathbf{x}^2 = (2/3, -1/3, 2/3), \quad \mathbf{x}^3 = (-1/3, 2/3, 2/3).$$

The linear orthogonal transformation

$$x_1 = \frac{2}{3}\xi_1 + \frac{2}{3}\xi_2 - \frac{1}{3}\xi_3, \quad x_2 = \frac{2}{3}\xi_1 - \frac{1}{3}\xi_2 + \frac{2}{3}\xi_3,$$

$$x_3 = -\frac{1}{3}\xi_1 + \frac{2}{3}\xi_2 + \frac{2}{3}\xi_3$$

reduces the quadratic form to the form

$$\lambda_1\xi_1^2 + \lambda_2\xi_2^2 + \lambda_3\xi_3^2 = 18\xi_1^2 + 9\xi_2^2 - 9\xi_3^2.$$

The equation of the surface relative to ξ_1, ξ_2, ξ_3 takes the form

$$18\xi_1^2 + 9\xi_2^2 - 9\xi_3^2 + 2\xi_1 + 2\xi_2 + 2\xi_3 + 1 = 0.$$

The canonical form of the surface $\left(u_1 = \xi_1 + \frac{1}{18}, \; u_2 = \xi_2 + \frac{1}{9}, \; u_3 = \xi_3 - \frac{1}{9}\right)$

$$-18u_1^2 - 9u_2^2 + 9u_3^2 = 17/18$$

is a hyperboloid of two sheets.

(b) The characteristic equation is $\lambda(\lambda-4)(\lambda-2) - 8(4-\lambda) = 0$, or $(4-\lambda)^2(\lambda+2) = 0$ (we expand the determinant in terms of the elements of the first column). The eigenvalues: $\lambda_1 = \lambda_2 = 4$, $\lambda_2 = -2$; the eigenvectors

$$\mathbf{x}^1 = \left(\frac{1}{\sqrt{5}}, \; 0, \; \frac{2}{\sqrt{5}}\right), \quad \mathbf{x}^2 = \left(-\frac{2}{\sqrt{30}}, \; -\frac{5}{\sqrt{30}}, \; \frac{1}{\sqrt{30}}\right),$$

$$\mathbf{x}^2 = \left(-\frac{2}{\sqrt{6}}, \; \frac{1}{\sqrt{6}}, \; \frac{1}{\sqrt{6}}\right).$$

Orthogonal transformation:

$$x_1 = \frac{1}{\sqrt{5}}\xi_1 - \frac{2}{\sqrt{30}}\xi_2 - \frac{2}{\sqrt{6}}\xi_3,$$

$$x_2 = 0\cdot\xi_1 - \frac{5}{\sqrt{30}}\xi_2 + \frac{1}{\sqrt{6}}\xi_3,$$

$$x_3 = \frac{2}{\sqrt{5}}\xi_1 + \frac{1}{\sqrt{30}}\xi_2 + \frac{1}{\sqrt{6}}\xi_3.$$

Canonical equation of the surface:

$$-2u_1^2 - 2u_2^2 + u_3^2 = 1$$

is a hyperboloid of revolution of two sheets.

315. Ellipse $\frac{y^2}{9} + \frac{z^2}{3} = 1$ with the semiaxes $a = 3$, $b = \sqrt{3}$ in the plane $x = 2$. Its vertices have the following coordinates in space:

$$(2, 3, 0), \quad (2, -3, 0), \quad (2, 0, \sqrt{3}), \quad (2, 0, -\sqrt{3}).$$

316. $x^2 + \frac{y^2}{16} - \frac{z^2}{16} = 1.$

317. (a) $\begin{cases} y^2 + z^2 = 2y - z, \\ x = 0, \end{cases}$

—the equation of the projection on the yz-plane. This is the equation of a circle.

(b) $\begin{cases} x^2 - 2xz + 5z^2 - 4x = 0, \\ y = 0, \end{cases}$

—the equation of the projection on the xz-plane. This is the equation of an ellipse ($AC - B^2 = 4 > 0$).

(c) $\begin{cases} x^2 + 4xy + 5y^2 - x = 0, \\ z = 0, \end{cases}$

—the equation of the projection on the xy-plane. This is also an ellipse.

318. Parabola: $\left(y - \frac{3}{2}\right)^2 = 3\left(z + \frac{1}{4}\right).$

319. $z = c$. *Hint.* Regard the equation of the surface as an implicit one:

$$F(x, y, z) \equiv \frac{x^2}{a^2} + \frac{z^2}{c^2} - \frac{y^2}{b^2} - 1 = 0.$$

Then the equation of the tangent plane at the point (x_0, y_0, z_0) has the form

$$\left(\frac{\partial F}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial F}{\partial y}\right)_0 (y - y_0) + \left(\frac{\partial F}{\partial z}\right)_0 (z - z_0) = 0.$$

321. (a) $x^2 + y^2 = 2z$—a paraboloid of revolution or an elliptic paraboloid,

(b) $\frac{x^2 + y^2}{a^2} + \frac{z^2}{c^2} = 1$—an ellipsoid of revolution.

322. (a) (3, 4, −2), (6, −2, 2). *Hint.* Pass to the parametric equations of the straight line.

(b) The straight line and the surface have no points in common.

324. $9X^2-16Y^2-16Z^2-90X+225=0$. *Hint.* By virtue of symmetry, it is clear that the directrix is a circle obtained in the section of the sphere by the plane $x=\alpha$. The value of α is found as the abscissa of the point of tangency of the straight line passing through the point S and touching the great circle $x^2+y^2=9$ in the xz-plane.

Chapter 4.

325. (a) $x^2+4y^2 \leqslant 1$—the interior of the ellipse with the semiaxes $a=1$, $b=1/2$ including its boundary (Fig. 66).

(b) $\frac{x^2}{9}-\frac{y^2}{4} \leqslant 1$—the region between the branches of the hyperbola with the semiaxes $a=3$, $b=2$ including the branches of the hyperbola (Fig. 67).

(c) $y^2 \geqslant 4x$—the exterior of the parabola, including the curve itself (Fig. 68).

(d) The entire plane except the origin (0, 0).

(e) $x+y>0$—the half-plane above the straight line $y=-x$ (Fig. 69).

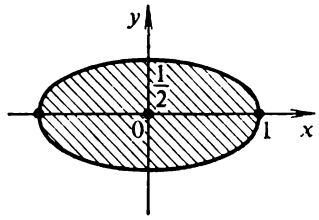

Fig. 66.

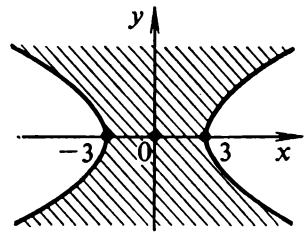

Fig. 67.

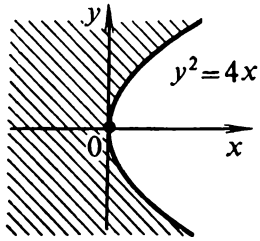

Fig. 68.

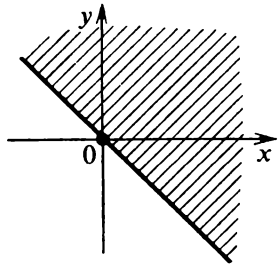

Fig. 69.

(f) $|y/x| \leqslant 1$, $x \neq 0$. The part of the plane adjoining to the x-axis between the straight lines $y=\pm x$, excepting the origin (Fig. 70).

326. (a) $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \leqslant 1$—the part of space found inside an ellipsoid with the semiaxes a, b, c including the surface of the ellipsoid (Fig. 71).

(b) $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2} \leqslant 1$—the part of space found inside a hyperboloid of one sheet, including its surface (Fig. 72).

(c) $x^2+y^2<2z$—the part of space found inside a paraboloid of revolution (Fig. 73).

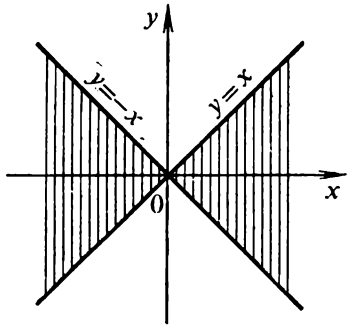

Fig. 70.

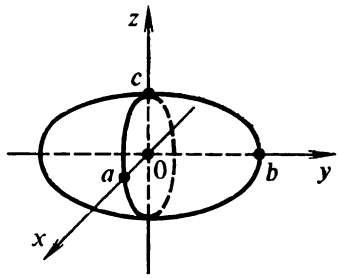

Fig. 71.

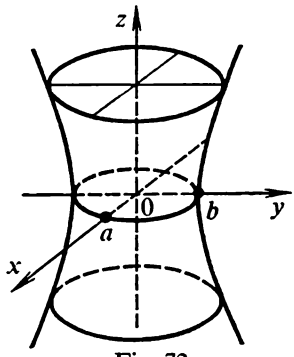

Fig. 72.

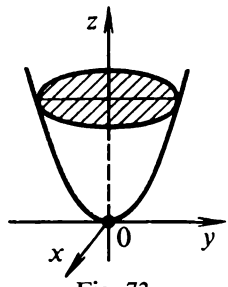

Fig. 73.

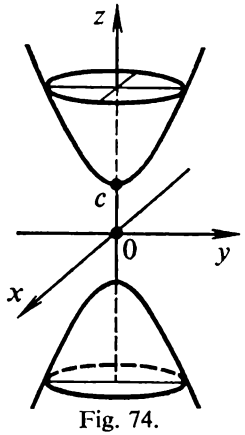

Fig. 74.

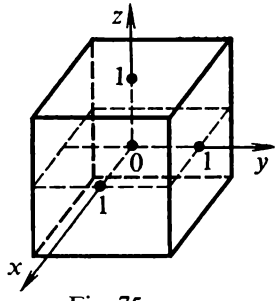

Fig. 75.

(d) $\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}\leqslant 1$—the part of space found outside a hyperboloid of two sheets, including its surface (Fig. 74).

(e) $|x|\leqslant 1, |y|\leqslant 1, |z|\leqslant 1$—the interior of the cube centred at the origin with edge equal to 2 including its faces. This cube is bounded by the planes $x=\pm 1, y=\pm 1, z=\pm 1$ (Fig. 75).

327. $f(1, 0) = 1$; $f(1, 1) = 2$; $f(2, 1) = 9/2$.

328. $f(x, y) = (x^2-y^2)/8$. *Hint.* Introduce new variables: $u = x+2y$, $v = x-2y$.

329. (a) $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1-c$—ellipses ($c < 1$); for $c = 1$—the origin; for $c > 1$—imaginary ellipses which means that the plane $u = c$ does not intersect the graph of the function;

(b) $y = cx^2$—parabolas with the y-axis as the axis of symmetry. For $c > 0$ the parabolas are found in the upper half-plane, and for $c < 0$—in the lower half-plane. For $c = 0$ we obtain the x-axis.

330. $\varrho = \sqrt{(1-2)^2+(0-1)^2+(1-0)^2} = \sqrt{3}$.

331. The point $M^0 = (0, 1)$; $\varrho(M^k, M^0) = \sqrt{\frac{1}{(1+k)^2}+\frac{1}{(k^2+1)^2}} \to 0$, $k \to \infty$.

332. All the points of the set $E_1 = \{|x| < 1, |y| < 1\} \subset E$ are interior.

333. (a) Yes, E is the interior of the square bounded by the straight lines $\pm y = \pm x+1$; (b) no, E is the interior of a hyperboloid of two sheets. Therefore it is impossible to connect two points found in the upper and lower sheets of the hyperboloid by a continuous curve belonging to E; (c) no.

334. (a) 2; (b) does not exist; consider the approaches to the point (0, 0) $x = y$; $x \neq 0$, $y = 0$.

335. $c = 0$. The limit of the function $\sqrt{1-x^2-4y^2}$ is equal to zero as the point (x, y) tends to the boundary of the ellipse $x^2+4y^2 = 1$.

336. (a) No; (b) the limit of the function in the direction of the vector $\omega = (\omega_1, \omega_2)$ is equal to $\frac{2\omega_1\omega_2}{\omega_1^2+\omega_2^2}$; therefore the function will be continuous at (0, 0) only in the direction of the vectors $\omega = (1, 0)$ and $\omega(0, 1)$, i.e. in the directions of the coordinate axes. Hence, this function is continuous at (0, 0) with respect to the variables x and y separately, and is not continuous with respect to both of them simultaneously.

337. *Hint.* The function $u = 1-x^2-y^2$ is continuous on the entire plane.

338. $u'_x = 3x^2$, $u'_y = 2y-2x$, $du = 3x^2\,dx+2(y-x)\,dy$.

339. $u'_x = 2xy^3$, $u'_y = 3x^2y^2$, $du = 2xy^3\,dx+3x^2y^2\,dy$.

340. $u'_x = \frac{1}{\sqrt{x^2+y^2}}$, $u'_y = \frac{y}{\sqrt{x^2+y^2}\,(x+\sqrt{x^2+y^2})}$,

$$du = \frac{dx}{\sqrt{x^2+y^2}}+\frac{y\,dy}{\sqrt{x^2+y^2}\,(x+\sqrt{x^2+y^2})}.$$

341. (a) $u'_x = \frac{-y}{x^2+y^2}$, $u'_y = \frac{x}{x^2+y^2}$, $du = \frac{-y\,dx+x\,dy}{x^2+y^2}$.

(b) $u'_x = y+\frac{1}{y}$, $u'_y = x-\frac{x}{y^2}$, $du = \left(y+\frac{1}{y}\right)dx+x\left(1-\frac{1}{y^2}\right)dy$;

(c) $u'_x = yx^{y-1}$, $u'_y = x^y \ln x$, $du = yx^{y-1}\,dx+x^y \ln x\,dy$;

(d) $u'_x \cosh(x+y)$, $u'_y = \cosh(x+y)$, $du = (dx+dy)\cosh(x+y)$;

(e) $u'_x = \sinh(x^2y+\sinh y)\cdot 2xy$, $u'_y = (x^2+\cosh y)\sinh(x^2y+\sinh y)$,

$du = [2xy\,dx+(x^2+\cosh y)\,dy]\sinh(x^2y+\sinh y)$.

342. (a) $\Delta = r$; (b) $\Delta = 4r\varphi-1$.

343. (a) $\dfrac{\partial u}{\partial t}=\dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial t}+\dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial t}=\dfrac{e^{t+\tau}}{2\sqrt{x+y}}+\dfrac{1}{2\sqrt{x+y}\cdot t}=$
$=\dfrac{e^{t+\tau}+t^{-1}}{2\sqrt{e^{t+\tau}+\ln t}}$; $\dfrac{\partial u}{\partial \tau}=\dfrac{1}{2\sqrt{x+y}}e^{t+\tau}=\dfrac{e^{t+\tau}}{2\sqrt{e^{t+\tau}+\ln t}}$; (b) $\dfrac{\partial u}{\partial t}=$
$=-y\sin(t+\tau)+x\cos(t-\tau)=-\sin(t-\tau)\sin(t+\tau)+\cos(t+\tau)\times$
$\times\cos(t-\tau)=\cos 2t$, $\dfrac{\partial u}{\partial \tau}=-y\sin(t+\tau)-x\cos(t-\tau)=-\cos 2\tau$.

344. (a) grad $u=\{2, 1\}$ (Fig. 76); (b) grad $u=\{4, -6\}$ (Fig. 77).

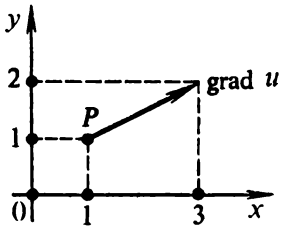

Fig. 76.

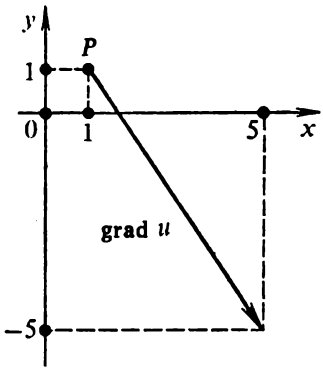

Fig. 77.

345. (a) $\dfrac{\partial u}{\partial \mathbf{n}}=(\operatorname{grad} u, \mathbf{n})=\dfrac{\sqrt{3}+1}{4}$; (b) $\dfrac{\partial u}{\partial \mathbf{n}}=(\operatorname{grad} u, \mathbf{n})=2\sqrt{3}-3$.

346. grad $u=\{4, -6\}$. The unit vector of this direction $\mathbf{n}_0=$
$=\left\{\dfrac{2}{\sqrt{13}}, \dfrac{-3}{\sqrt{13}}\right\}$, $\dfrac{\partial u}{\partial \mathbf{n}_0}=4\dfrac{2}{\sqrt{13}}-6\cdot\dfrac{-3}{\sqrt{13}}=\dfrac{26}{\sqrt{13}}=2\sqrt{13}$. We may write at once that $\dfrac{\partial u}{\partial \mathbf{n}_0}=|\operatorname{grad} u|=2\sqrt{13}$. This is the maximum directional derivative.

347. (a) Let α and β be angles formed by the gradient of the function with the x- and y-axes, respectively, $\cos\alpha=\dfrac{u'_x(P)}{|\operatorname{grad} u(P)|}=\dfrac{\sqrt{3}}{2}$, $\cos\beta=$
$=\dfrac{u'_y(P)}{|\operatorname{grad} u(P)|}=\dfrac{1}{2}$; $\alpha=\pi/6$, $\beta=\pi/3$; (b) $\alpha=\pi/3$, $\beta=\pi/6$.

348. (a) $u''_{x^2}=2\dfrac{y-x^2}{(x^2+y)^2}$, $u''_{y^2}=\dfrac{-1}{(x^2+y)^2}$, $u''_{xy}=\dfrac{-2x}{(x^2+y)^2}$,

$$d^2u=[2(y-x^2)\,dx^2-4x\,dx\,dy-dy^2]/(x^2+y)^2.$$

(b) $u''_{x^2}=\dfrac{-y^2}{(2xy+y^2)^{3/2}}$, $u''_{xy}=\dfrac{xy}{(2xy+y^2)^{3/2}}$, $u''_{y^2}=\dfrac{-x^2}{(2xy+y^2)^{3/2}}$,

$$d^2u=-(y\,dx-x\,dy)^2/(2xy+y^2)^{3/2}.$$

351. (a) $\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial^2 f}{\partial \xi^2}a^2$, $\dfrac{\partial^2 u}{\partial x\,\partial y}=ab\dfrac{\partial^2 f}{\partial \xi\,\partial \eta}$, $\dfrac{\partial^2 u}{\partial y^2}=b^2\dfrac{\partial^2 f}{\partial \eta^2}$,

$$d^2u=a^2f''_{\xi^2}\,dx^2+2abf''_{\xi\eta}\,dx\,dy+b^2f''_{\eta^2}\,dy^2;$$

(b) $\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial^2 f}{\partial \xi^2}+2\dfrac{\partial^2 f}{\partial \xi\,\partial \eta}+\dfrac{\partial^2 f}{\partial \eta^2}$, $\dfrac{\partial^2 u}{\partial x\,\partial y}=\dfrac{\partial^2 f}{\partial \xi^2}-\dfrac{\partial^2 f}{\partial \eta^2}$,

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 f}{\partial \xi^2}-2\frac{\partial^2 f}{\partial \xi\,\partial \eta}+\frac{\partial^2 f}{\partial \eta^2},$$

$$d^2u=\frac{\partial^2 f}{\partial \xi^2}(dx+dy)^2+2\frac{\partial^2 f}{\partial \xi\,\partial \eta}(dx^2-dy^2)+\frac{\partial^2 f}{\partial \eta^2}(dx-dy)^2.$$

352. (a) $u-5=2(x-1)+4(y-2)$, $\dfrac{x-1}{2}=\dfrac{y-2}{4}=\dfrac{u-5}{-1}$;

(b) $u-1=2(x-2)+2(y+1)$, $\dfrac{x-2}{2}=\dfrac{y+1}{2}=\dfrac{u-1}{-1}$.

353. (a) by Taylor's formula,

$$\Delta u=u(1+h,\,2+k)-u(1,\,2)=du+\frac{1}{2}\,d^2u=4h-3k+h^2-k^2+kh$$

(the derivatives of order higher than 2 are equal to zero);

(b) $\Delta u=2h+k+h^2+2hk+h^2k$.

354. $y+xy+\dfrac{1}{3!}(3x^2y-y_3)$. *Remark.* It is possible to take advantage of one-dimensional Taylor's formulas for the functions

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots,\qquad \sin y=y-\frac{y^3}{3!}+\ldots\ .$$

355. $1+\dfrac{[(x-1)+(y+1)]}{1!}+\dfrac{[(x-1)^2+2(x-1)(y+1)+(y+1)^2]}{2!}$

$$+\frac{[(x-1)^3+3(x-1)^2(y+1)+3(x-1)(y+1)^2+(y+1)^3}{3!}$$

$$\equiv 1+\frac{(x+y)}{1!}+\frac{(x+y)^2}{2!}+\frac{(x+y)^3}{3!}.$$

356. (a) $\theta=1/2$; (b) $3\theta^2+2\theta=2$, $\theta=(\sqrt{7}-1)/3$.

357. (2, 0)—a stationary point; $z''_{x^2}=2$, $z''_{y^2}=4$, $z''_{xy}=0$; $a_{11}=z''_{x^2}$ (2, 0), $a_{22}=z''_{y^2}$ (2, 0), $a_{12}=z''_{xy}$ (2, 0), $a_{11}a_{22}-a_{12}^2=8>0$, $a_{11}=2>0$, hence, at the point (2, 0) the function has a minimum, $z_{\min}=0$.

358. Stationary point (2, 0); $a_{11}a_{22}-a_{12}^2=-8$, no extremum.

359. Stationary point (0, 0); $a_{11}=0$, $a_{22}=-4$, $a_{12}=4$, $a_{11}a_{22}-a_{12}^2=-16<0$, no extremum.

360. Stationary points: (0, 0), $(\pm\sqrt{2},\ \mp\sqrt{2})$; $z''_{x^2}=12x^2-4$, $z''_{y^2}=12y^2-4$, $z''_{xy}=4$; $z''_{x^2}(\pm\sqrt{2},\ \mp\sqrt{2})=20$, $z''_{y^2}(\pm\sqrt{2},\ \mp\sqrt{2})=20$, $a_{11}a_{22}-a_{12}^2=396>0$. At the points $(\sqrt{2},\ -\sqrt{2})$, $(-\sqrt{2},\ \sqrt{2})$—a local minimum. At the point (0, 0) $a_{11}=-4$, $a_{22}=-4$, $a_{12}=4$, $a_{11}a_{22}-a_{12}^2=0$. The question on extremum remains undecided. Investigating the increment of the function on the straight lines $y=0$ and $y=x$, we make sure that there is no extremum at the point (0, 0).

361. $u_{\min}=-4/3$ for $x=-2/3$, $y=-1/3$, $z=1$.

362. (a) The function has no greatest value; $\sup z=2$. The function is discontinuous.

(b) Has. The domain of definition $|x| \leqslant 1$, $|y| \leqslant 1$ is closed and the function z is continuous on that domain, therefore it has the greatest value; $(0, \pm 1)$— a stationary point; $z(0, \pm 1) = \pm 1/2$; at the boundary of the square at the points $x = \pm 1$, $y = \sqrt{3} - 1$ the function reaches its greatest value equal to $(1+\sqrt{3})/4$.

363. $F(x, y) \equiv \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0$, $F'_x = \dfrac{2x}{a^2}$, $F'_y = \dfrac{2y}{b^2}$,

$$\frac{dy}{dx} = -\frac{F'_x}{F'_y} = -\frac{b^2 x}{a^2 y}, \quad \frac{d^2y}{dx^2} = -\frac{b^4}{a^2 y^3}.$$

364. $\dfrac{\partial z}{\partial x} = \dfrac{z \sin x - \cos y}{\cos x - y \sin z}$, $\dfrac{\partial z}{\partial y} = \dfrac{x \sin y - \cos z}{\cos x - y \sin z}$.

366. $\dfrac{\partial u}{\partial x} = \dfrac{\partial \psi}{\partial v} \Big/ \dfrac{D(\varphi, \psi)}{D(u, v)}$, $\dfrac{\partial v}{\partial x} = -\dfrac{\partial \psi}{\partial u} \Big/ \dfrac{D(\varphi, \psi)}{D(u, v)}$,

$$\frac{\partial u}{\partial y} = -\frac{\partial \varphi}{\partial v} \Big/ \frac{D(\varphi, \psi)}{D(u, v)}, \quad \frac{\partial v}{\partial y} = \frac{\partial \varphi}{\partial u} \Big/ \frac{D(\varphi, \psi)}{D(u, v)},$$

where $\dfrac{D(\varphi, \psi)}{D(u, v)} = \begin{vmatrix} \varphi'_u & \varphi'_v \\ \psi'_u & \psi'_v \end{vmatrix}$.

367. $\dfrac{\partial z}{\partial x} = -\dfrac{c}{u} \sin v$, $\dfrac{\partial z}{\partial y} = \dfrac{c}{u} \cos v$. *Hint.* $\dfrac{\partial z}{\partial x} = c \dfrac{\partial v}{\partial x}$

The derivative $\dfrac{\partial v}{\partial x}$ is found from the first two equations by regarding u, v as implicit functions of x and y.

368. (a) $x + 6\sqrt{3}z - 37\sqrt{3} = 0$; (b) $\dfrac{x_0 x}{a^2} + \dfrac{y_0 y}{b^2} + \dfrac{z_0 z}{c^2} = 1$.

369. $x + 4y + 6z = \pm 21$.

370. $x + z + a = 0$, $\dfrac{x}{1} = \dfrac{y-a}{0} = \dfrac{z+a}{1}$.

371. A square ($S = xy$, $2x + 2y = l$, Lagrange's function $L = xy + \lambda(2x + 2y - l)$).

372. Method 1. Reduce the equation of the ellipse to the canonical form. The eigenvalues $\lambda_1 = 9$, $\lambda_2 = 1$; $9\xi^2 + \eta^2 = 9$, $\xi^2 + \dfrac{\eta^2}{9} = 1$. Hence, the semiaxes of the ellipse: $a = 1$, $b = 3$; $2a = 2$, $2b = 6$.

Method 2. The given ellipse is situated symmetrically about the origin, therefore the square of the distance from the origin to the point of the ellipse (x, y) equal to $x^2 + y^2$ attains the greatest (least) value when the point (x, y) falls on the major (minor) axis of the ellipse. Therefore it is necessary to investigate the function $u = x^2 + y^2$ for a conditional extremum with the restraint $5x^2 + 8xy + 5y^2 = 9$.

373. An equilateral triangle. *Hint.* $S = \sqrt{l(l-x)(l-y)(l-z)}$, $x + y + z = 2l$, x, y, z being the sides of the triangle. We substitute the value $l - z = x + y - l$ into S, and investigate the obtained function for an ordinary extremum. We may also investigate the problem by writing Lagrange's function:

$$L(S, \lambda) = S + \lambda(x + y + z - 2l).$$

Chapter 5.

374. (a) $S = 1$. *Hint.* $\dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$.

(b) $S = \dfrac{13}{36}$. *Hint.* $\dfrac{1}{(n+1)(n+4)} = \dfrac{1}{3}\left(\dfrac{1}{n+1} - \dfrac{1}{n+4}\right)$,

$$S_N = \sum_{n=1}^{N} \frac{1}{(n+1)(n+4)} = \frac{1}{3}\left[\frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{N+2} - \frac{1}{N+3} - \frac{1}{N+4}\right].$$

(c) $S = 1/4$. *Hint.*

$$\frac{1}{n(n+1)(n+2)} = \frac{1}{2}\left[\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right].$$

375. *Hint.* When proving the divergence of the harmonic series by Cauchy's test, consider

$$|S_{2n} - S_n| = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}.$$

377. Diverges $\left(\dfrac{n+1}{2n+1} \to \dfrac{1}{2} \neq 0\right)$.

378. Diverges $\dfrac{1}{\sqrt{n^2+2n}} \geqslant \dfrac{1}{\sqrt{n^2+2n^2}} = \dfrac{1}{n\sqrt{3}}$.

379. (a) Converges, $\sqrt{\dfrac{n}{n^4+1}} \leqslant \dfrac{1}{n^{3/2}}$; (b) converges, $\dfrac{1}{k^2+1} \leqslant \dfrac{1}{k^2}$.

380. (a) Diverges; (b) and (c) converge. *Hint.* Apply theorem 1 of Sec. 9.4 from [1].

381. Converges. **382.** Converges. **383.** Converges. **384.** Converges. **385.** Diverges.

386. Converges for $\varepsilon > 1$; diverges for $0 < \varepsilon \leqslant 1$. **387.** Converges.

388. (a) Converges, $u_n \leqslant \dfrac{2}{3n^{3/2}}$; (b) converges, $u_n \leqslant \int_0^{1/n} x^3\,dx = \dfrac{1}{4n^4}$. **389.** Converges conditionally. **390.** Converges absolutely. **391.** *Hint.* $|a_k b_k| \leqslant \dfrac{a_k^2 + b_k^2}{2}$. **392.** (a) $x > 1$; (b) $x > 0$; (c) $-\infty < x < \infty$; (d) $-1 < x < 0$, $0 < x < 1$.

393. (a) Converges uniformly to zero; (b) converges uniformly to zero; (c) converges uniformly to zero; (d) converges nonuniformly to zero $\left(\max\limits_{0 \leqslant x \leqslant 1} f_n(x) = \dfrac{1}{4} \not\to 0\right)$.

395. (a) The given series is convergent for $-1 \leqslant x < 1$. The differentiated series $\sum\limits_{1}^{\infty} x^{n-1}$ converges uniformly on the set $[-\delta, \delta]$ for any $\delta < 1$. Therefore termwise differentiation is valid on the indicated set. Let $S(x) = \sum\limits_{1}^{\infty} x^n/n$, then $S'(x) = \sum\limits_{1}^{\infty} x^{n-1} = 1/(1-x)$. Integrating, we obtain

$$S(x)=\int_0^x \frac{dt}{1-t}=-\ln(1-t)\Big|_0^x=-\ln(1-x) \quad (-1\leqslant x<1).$$

(b) The given series is uniformly convergent for $|x|\leqslant\delta<1$, which can be checked by D'Alembert's test. Therefore it can be integrated termwise:

$$\int_0^x S(t)\,dt=\left[t+t^2+\ldots+t^{n+1}+\ldots\right]_0^x=\sum_{n=1}^{\infty}x^n=\frac{x}{1-x} \quad (|x|\leqslant\delta<1).$$

Differentiating the last equality with respect to x, we get

$$S(x)=\frac{1}{(1-x)^2} \quad (|x|<1); \qquad \text{(c) } S(x)=\frac{1-x^2}{(1+x^2)^2} \quad (|x|<1).$$

396. (a) $R=1$, $(-1, 1)$, for $x=\pm1$ the series converges; (b) $R=0$, $x=0$; c) $R=1/3$, $(-1/3, 1/3)$, for $x=\pm1/3$ the series diverges.

397. (a) $x+\frac{x^3}{3}+\ldots$; (b) $x-\frac{x^3}{3}+\ldots$; (c) $e\left(1-\frac{x^2}{2}+\ldots\right)$.

398. (a) $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)n!}$ $(-\infty<x<\infty)$;

(b) $x-\frac{x^3}{3^2}+\frac{x^5}{5^2}-\ldots+(-1)^{n+1}\frac{x^{2n+1}}{(2n+1)^2}+\ldots$ $(-1\leqslant x\leqslant1)$

Hint. Expand the function arc tan x into Taylor's series in powers of x.

399. 32.831. **400.** $e^{\mp2}\left[1+\sum_{n=1}^{\infty}\frac{(x\pm2)^n}{n!}\right]$ $(|x|<\infty)$. *Hint.* $e^x=e^{\mp2}e^{x\pm2}$.

Chapter 6.

401. (a) $y-2xy'=0$; (b) $y''=0$; (c) $y'=y$; (d) $x+yy'=0$; (e) $y''-y'-2y=0$.

402. (a) Isoclines are straight lines $x=k$ (straight lines parallel to the y-axis) (Fig. 78). The exact solution of the equation is $y=\frac{x^2}{2}+C$.

(b) $1+y^2=k$—isoclines, $k\geqslant1$. These are straight lines parallel to the x-axis (Fig. 79).

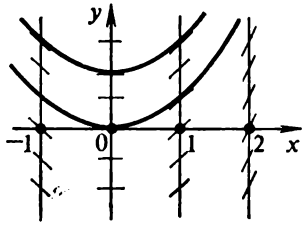

Fig. 78.

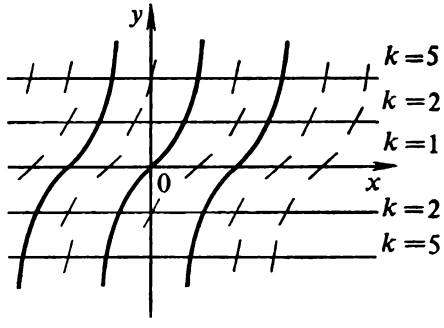

Fig. 79.

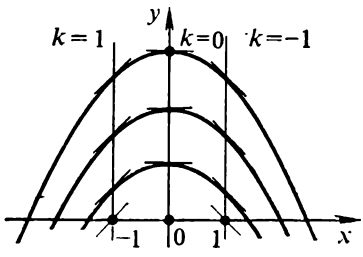

Fig. 80.

(c) isoclines: $x = -k$ (Fig. 80).

403. $y = C(x+1)e^{-x}$, $x = -1$.

404. $\ln|x| = C+\sqrt{y^2+1}$.

405. $1-e^{-t} = Ce^{t}$.

406. $y = -2/(1+C\exp(-x^2))$.

407. $y = (x-2)^3$. **408.** $y = 2-\cos x$.

409. $y = Cx^3$, $3y = xy'$ is the differential equation of the family of curves.

410. 0.5 kg. *Hint.* Write the differential equation of our problem. Let $y(t)$ be the amount of salt in the tank at time t. Let us find out the change in salt content during the time Δt (from the instant t to $t+\Delta t$). Since, by hypothesis, 5 litres of unsalted water is supplied per minute, during the time Δt, $5\,\Delta t$ litres of water will be supplied. This quantity of water will contain $5\,\Delta t \cdot 0 = 0$ kg of salt. One litre of the solution contains $\frac{y(t)}{100}$ kg of salt, hence, the salt content in the flowing out mixture will amount approximately (with an accuracy up to an infinitesimal of higher order than Δt) to

$$5\,\Delta t \cdot \frac{y(t)}{100}\,\text{kg} = 0.05\,\Delta t\, y(t)\,\text{kg}.$$

Thus, the solution, inflowing during the time Δt, contains 0 kg of salt, while the solution, flowing out of the tank during the same time, contains $0.05\,\Delta t\, y(t)$ kg of salt. The increment of the amount of salt during this time

$$y(t+\Delta t)-y(t) \approx 0-0.05\,\Delta t\, y(t).$$

Dividing by Δt and passing to the limit as $\Delta t \to 0$, we obtain the differential equation

$$y'(t) = -0.05\,y(t).$$

The general solution of this equation has the form $y = C\exp(-0.05t)$. By hypothesis, $y(0) = 10$ kg, hence, $C = 10$. In $t = 1$ hour $= 60$ min we obtain (when deriving the equation, we supposed the change in time to be in minutes)

$$y(60) = 10\exp(-3) \approx 1/2\ \text{kg}.$$

411. $t_0 = 40$ min. *Hint.* If $\theta(t)$ is the temperature of the body at time t, then the differential equation of the problem will be written as

$$\frac{d\theta}{dt} = -k[\theta(t)-20].$$

The general solution of this equation $\theta(t) = 20+C\exp(-kt)$. The constant C and the proportionality factor k are found from the conditions: $\theta(0) = 100$, $\theta(10) = 60$, $C = 80$, $k = 0.1\ln 2$.

Further, $\theta(t_0) = 25$, i.e. $25 = 20+80\exp(-0.1t_0\ln 2)$.

412. $x^2+C(y+x) = 0$; $x = 0$, **413.** $y = 0$; $x(y-x) = Cy$.

414. $y = C\exp(y/x)$.

415. The general solution: $x^3(y+C) = Cy$.

416. $x = C\exp\left(\frac{2x^3}{3x^2-2y}\right)$.

417. The equation is reduced to the form

$$y' = x\left(\cos^2\frac{y}{x^2}+\frac{2y}{x^2}\right),$$

i.e. $\alpha = 2$, $f(t) = \cos^2 t+2t$. The solution is carried out by the substitution $y = tx^2$. We may also take advantage of the formula derived in [3], Sec. 1.3, (7):

$$x = C\exp\left[\int\frac{dt}{f(t)-2t}\right] = C\exp\left[\int\frac{dt}{\cos^2 t}\right] = C\exp(\tan t)$$

$$= C\exp\left(\tan\frac{y}{x^2}\right).$$

418. *Hint.* Set up the differential equation of the problem (see Fig. 81). Let $M = (x, y)$ be the point of tangency; MN tangent line; $ON \perp MN$; by hypothesis, $ON = OP = |x|$. If

$$Y = y = y'(X-x)$$

is the equation of the tangent, then

$$ON = \frac{|-y+y'x|}{\sqrt{1+y'^2}}$$

is the distance of the point (0, 0) from the straight line MN. Thus,

$$|x| = \frac{|xy'-y|}{\sqrt{1+y'^2}},$$

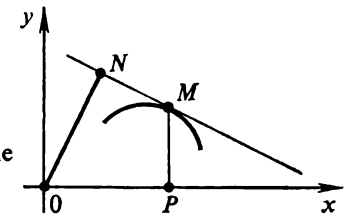

Fig. 81.

whence

$$(x^2-y^2)\,dx+2xy\,dy = 0.$$

This is a homogeneous equation. Its general solution: $Cx = y^2+x^2$.

419. The equation is reduced to the form

$$y' = \frac{1}{x^2}(4-xy-y^2x^2) = \frac{1}{x^2}f(xy),$$

i.e. $\alpha = -1$, $f(t) = 4-t-t^2$. The general solution:

$$y = \frac{2}{x}+\frac{4}{Cx^5-x}.$$

420. (a) The equation is reduced to the form

$$y' = \frac{-1}{3x^2}(2+x^2y^2),$$

i.e. $\alpha = -1$, $f(t) = -\frac{1}{3}(2+t^2)$. The equation can be solved by the substitution $xy = t$ or by the formula (7) of Sec. 1.3, [3]:

$$x = C\exp\left(\int\frac{dt}{f(t)+t}\right), \quad x = C\left(\frac{t-1}{t-2}\right)^3 = C\left(\frac{xy-1}{xy-2}\right)^3,$$

(put $\bar{C}^3C = 1$)

$$y = \frac{2\bar{C}x^{1/3}-1}{x(\bar{C}x^{1/3}-1)} = \frac{1}{x}+\frac{\bar{C}x^{1/3}}{x(\bar{C}x^{1/3}-1)} = \frac{1}{x}+\frac{1}{C^*x^{2/3}+x} \quad \left(C^* = -\frac{1}{C}\right).$$

(b) Substitution: $yx^{-\alpha} = t$.

421. $y = Ce^{-2x}+2x-1$. **422.** $y = Cx^2+x^4$. **423.** $y = e^x(\ln|x|+C)$.

424. $y = \frac{e^x}{x}+\frac{1-e}{x}$. **425.** $y = x(C+\sin x)$.

426. $x = \sin y(C-\cos y)$. The equation is linear with respect to the function $x = x(y)$:

$$\frac{dx}{dy} = \sin^2 y + x\cot y.$$

427. $y = 0$; $y^3 = -1/[3\cos^3 x(C+\tan x)]$. **428.** $y^2 = Cx^2-2x$. **429.** $y = 0$; $y = x^4\ln^2|Cx|$. **430.** $y = 0$; $y = 1/(x^2+Cx)$.

431. (a) Yes. All distance axioms are readily checked.

(b) Yes. The first and second axioms are obvious. Check the triangle inequality: $\varrho(x, y) \leqslant \varrho(x, z)+\varrho(z, y)$.

If $x = y = z$, then $0 \leqslant 0+0$; if $x = y$, $z \neq x$, then $0 \leqslant 1+1 = 2$; if $x \neq y$, $x = z$, then $1 \leqslant 0+1 = 1$; if $x \neq y$, $x \neq z$, $y \neq z$, then $1 \leqslant 1+1 = 2$.

432. Yes. The first axiom: if $f(x) \equiv g(x)$, then $\varrho(f, g) = 0$. Conversely, let $\varrho(f, g) = 0$. Then

$$\int_a^b [f(x)-g(x)]^2\,dx = 0.$$

Since $[f(x)-g(x)]^2$ is a nonnegative continuous function on $[a, b]$, we have $[f(x)-g(x)]^2 \equiv 0$, i.e. $f(x) \equiv g(x)$ (see Sec. 6.2, theorem 8, [1]). The second axiom: $\varrho(f, g) = \varrho(g, f)$ is obvious. The third axiom: we have

$$\int_a^b [f(x)+\lambda g(x)]^2\,dx \geqslant 0$$

for $\forall\lambda$, i.e. a quadratic trinomial with respect to λ

$$\int_a^b f^2(x)\,dx+2\lambda\int_a^b f(x)\,g(x)\,dx+\lambda^2\int_a^b g^2(x)\,dx \geqslant 0.$$

This is possible if the discriminant of the equation is nonpositive:

$$\left(\int_a^b f(x)\,g(x)\,dx\right)^2-\int_a^b f^2(x)\,dx\int_a^b g^2(x)\,dx \leqslant 0,$$

whence

$$\left|\int_a^b f(x)\,g(x)\,dx\right| \leqslant \left(\int_a^b f^2(x)\,dx\right)^{1/2}\left(\int_a^b g^2(x)\,dx\right)^{1/2}$$

(*Buniakowski's inequality* for integrals, see also [3], Sec. 4.8). Further, applying Buniakowski's inequalty, we have

$$\int_a^b [f(x)+g(x)]^2 dx$$

$$\leqslant \int_a^b |f(x)+g(x)|\,|f(x)|\,dx + \int_a^b |f(x)+g(x)|\cdot|g(x)|\,dx$$

$$\leqslant \left(\int_a^b [f(x)+g(x)]^2\,dx\right)^{1/2}\left[\left(\int_a^b f^2(x)\,dx\right)^{1/2}+\left(\int_a^b g^2(x)\,dx\right)^{1/2}\right]$$

or

$$\left(\int_a^b [f(x)+g(x)]^2\,dx\right)^{1/2} \leqslant \left(\int_a^b f^2(x)\,dx\right)^{1/2}+\left(\int_a^b g^2(x)\,dx\right)^{1/2}$$

(*Minkowski's inequality* for integrals). Now, by Minkowski's inequality, we obtain

$$\varrho(f, g) = \left(\int_a^b [f(x)-g(x)]^2\,dx\right)^{1/2}$$

$$= \left(\int_a^b \{[f(x)-\varphi(x)]+[\varphi(x)-g(x)]\}^2\,dx\right)^{1/2}$$

$$\leqslant \left(\int_a^b [f(x)-\varphi(x)]^2\,dx\right)^{1/2}+\left(\int_a^b [\varphi(x)-g(x)]^2\,dx\right)^{1/2} = \varrho(f, \varphi)+\varrho(\varphi, g)$$

for any function $\varphi(x)$ continuous on $[a, b]$.

433. $\alpha < 1/2$. *Hint.* $\varrho(f_n, 0) = \left(\int_0^{1/n} n^{2\alpha}\,dx\right)^{1/2} = n^{\alpha-\frac{1}{2}}$.

434. No. The fundamental sequence $\left\{3-\frac{1}{n}\right\}$ converges to the number 3 which does not belong to M.

435. (a) Yes. $\varrho(F(x), F(y)) = |F(x)-F(y)| = |x^2-y^2| = |x-y|\cdot|x+y| \leqslant$ $\leqslant |x-y|\,(|x|+|y|) \leqslant \frac{2}{3}|x-y| = \alpha\varrho(x, y)$, where $\alpha = 2/3 < 1$.

(b) No. If $x = 1$, $y = 0$, then $|F(x)-F(y)| = 1 = 1\,|x-y|$ $(\alpha = 1)$.

436. $x_0 = 1/2$, $x_1 = F(x_0) = 1/2^2$, $x_2 = 1/2^4$, ..., $x_n = 1/2^{2^n}$, ...

437. (a) $\bar{x} = 2/(1+\sqrt{5})$. Yes, since

$$\max_{1/2 \leqslant x \leqslant 1} |F'(x)| = \max_{1/2 \leqslant x \leqslant 1} \frac{1}{(1+x)^2} = \frac{4}{9} < 1.$$

(b) The axis $x_2 = 0$; (c) the straight lines $x_2 = 0$ and $x^2 = 1$.

438. (a) $\delta < 1/36$. *Hint.* According to the existence theorem,

$$\delta < \min \{a, 1/N, b/M\},$$

where

$$N = \sup_D \left| \frac{\partial f}{\partial y} \right|, \quad M = \max_D |f(x, y)|.$$

In this case $M = 36$, $N = 24$. The solution of this problem has the form $y = 2/(3-2x^2)$. Thus, as $x \to \sqrt{3/2}$ the solution $y(x) \to \infty$. Hence, in fact the solution exists in the interval $(0, \sqrt{3/2})$, which exceeds the interval $(1-\delta, 1+\delta)$ $(\delta < 1/36)$.

Note that within the entire interval $(0, 2) = (1-a, 1+a)$ the solution of this problem (with the indicated initial conditions!) does not exist.

(b) $\delta < a = \sqrt{2}/4$. In the present case $a = \dfrac{b}{M} = \dfrac{1}{N} = \dfrac{1}{8a}$. Thus, the solution exists in a minimum possible interval $[-\delta, \delta]$ $(\delta < a)$, i.e. the existence theorem yields a nonimprovable result in terms of the size of an interval, where the solution for the given right-hand side of $f(x, y)$.

439. $y(1) \approx 1.248$. *Hint.* When equations are solved approximately, it is always recommended to determine the interval $(x_0-\delta, x_0+\delta)$, where the solution $y(x)$ exists. The number δ is found from the existence theorem. If the point at which we are interested in the value of the solution belongs to the indicated interval, then we may apply Euler's method. This problem illustrates the application of this method. The equation can be solved. Its solution satisfying the initial condition, has the form $y = \exp(x^2/4)$, i.e. the solution exists throughout the real axis and, therefore, we may apply Euler's method without any limitations. By Euler's method,

$$y(1) \approx y_0 + h \sum_{k=0}^{9} f(x_k, y_k)$$

where $x_0 = 0$, $x_1 = 0.1, \ldots, x_9 = 0.9$, $x_{10} = 1$; $y_0 = 1$, $y_1 = y_0 + hf(x_0, y_0), \ldots, y_9 = y_8 + hf(x_8, y_8), y_{10} = y_9 + hf(x_9, y_9)$. Hence, $y(1) \approx y_{10}$. All these computations can be tabulated:

k	x_k	y_k	$hf(x_k, y_k)$
0	0	1	0
1	0.1	1	0.005
2	0.2	1.005	0.010
3	0.3	1.015	0.015
4	0.4	1.030	0.021
5	0.5	1.051	0.026
6	0.6	1.077	0.032
7	0.7	1.109	0.039
8	0.8	1.148	0.046
9	0.9	1.194	0.054
10	1	1.248	—

The true value of the solution $y(1) = \exp(1/4) \approx 1.284$, i.e. we have obtained an approximate value of the solution with the correct first decimal digit.

440. $y(2) \approx 4.781$ (the exact value of $y(2) = 3(e-1)$). *Hint.* The solution of the equation exists throughout the number axis. The general solution of the equation: $y = Ce^x - x - 1$.

441. (a) $y = C \exp(\pm x)$ (Fig. 82). No singular solutions.

(b) $y(x+C)^2 = 1$, $y = 0$ (Fig. 83). No singular solutions.

(c) $(x+C)^2 + y^2 = 1$, $y = \pm 1$ are singular solutions (Fig. 84). At each of their points the integral curves $y = \pm 1$ are touched by one more integral curve (by a circle).

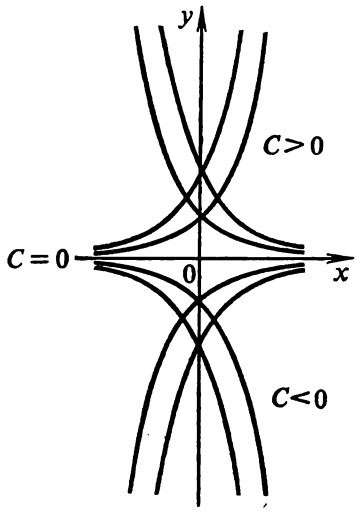

Fig. 82.

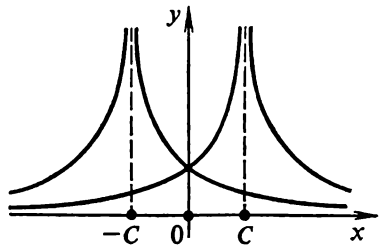

Fig. 83.

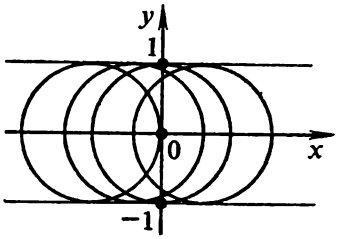

Fig. 84.

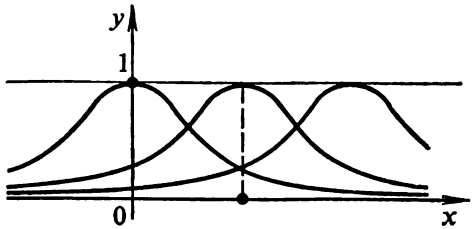

Fig. 85.

(d) $y[1+(x-C)^2] = 1$; $y = 0$; $y = 1$ is a singular solution (Fig. 85). Note that $y = 0$ is not a singular solution. This integral curve is not touched by other integral curves.

(e) $x^2 + C^2 = 2Cy$—parabolas; $y = \pm x$—singular solutions (Fig. 86).

(f) $(Cx+1)^2 = 1 - y^2$—ellipses; $y = \pm 1$—singular solutions (Fig. 87).

Hint. Singular solutions can be sought for in different ways. For instance, in (e), solving the equation with respect to y, we obtain the homogeneous equation

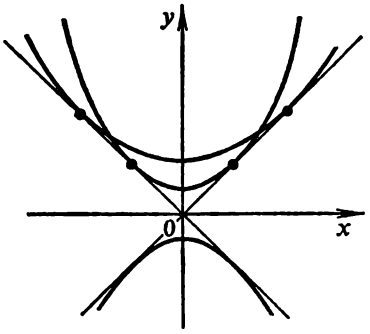

Fig. 86.

$y' = \frac{y \pm \sqrt{y^2-x^2}}{x} (x \neq 0)$. If the partial derivative with respect to y of the right-hand member of the last equation becomes infinite along a smooth curve, then this curve can be a singular solution. In the present case

$$\frac{\partial}{\partial y}\left[\frac{y \pm \sqrt{y^2-x^2}}{x}\right] = \frac{1}{x}\left[1 \pm \frac{y}{\sqrt{y^2-x^2}}\right] = \infty$$

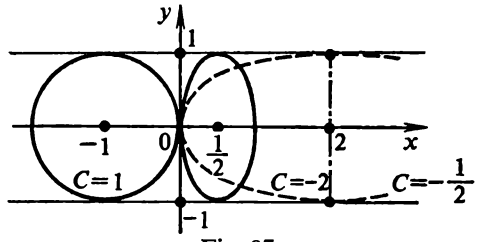

Fig. 87.

for $y = \pm x$. By checking, we make sure that $y = \pm x$ are solutions of our equation. It can be readily ascertained that these straight lines are touched at every point by one more integral curve of the family $x^2+C^2 = 2Cy$. Hence, $y = \pm x$ are singular solutions.

The same solutions can be found from the system

$$\left.\begin{aligned} x^2+C^2-2Cy &= 0, \\ -\frac{\partial}{\partial C}(x^2+C^2-2Cy) &= 0, \end{aligned}\right\} \qquad \left.\begin{aligned} (x^2+C^2-2Cy) &= 0, \\ C-y &= 0. \end{aligned}\right\} \; y = \pm x.$$

The further investigation is carried out as above.

442. (a) The given equation does not contain the variable y explicitly. Introducing the parameter: $\frac{dy}{dx} = p$, $x = p^3+p$, $dy = p\,dx = p(3p^2+1)\,dp$; we have

$$\begin{cases} x = p^3+p, \\ y = \frac{3}{4}p^4+\frac{p^2}{2}+C \end{cases}$$

which is the parametric representation of the solution.

(b) The given equation does not contain the variable x explicitly. Introducing the parameter $\frac{dy}{dx} = p$, $y = p^2+2p^3$, $dx = \frac{dy}{p} = (2+6p)\,dp$, we obtain the solution

$$\begin{cases} x = 2p+3p^2+C, \\ y = p^2+2p^3, \end{cases}$$

represented parametrically; $y = 0$ is also a solution of the equation.

(c) The given equation does not contain y either. The parameter p can be introduced by the formula $\frac{dy}{dx}=p$. Then $x=p\sqrt{1+p^2}$, $dy=p\,dx=$ $=p\left(\sqrt{1+p^2}+\frac{p^2}{\sqrt{1+p^2}}\right)dp$, $3y=(2p^2-1)\sqrt{1+p^2}+C$. Here the parameter can also be introduced by the formula $\frac{dy}{dx}=\sinh p$, $x=\sinh p\sqrt{1+\sinh^2 p}=$ $=\frac{1}{2}\sinh 2p$, $dy=\sinh p\cdot dx=\sinh p\cdot\cosh 2p\,dp=(2\cosh^2 p-1)d\cosh p$, $y=\frac{2}{3}\cosh^3 p-\cosh p+C$.

$$\begin{cases} x=p\sqrt{1+p^2}, \\ 3y=(2p^2-1)\sqrt{1+p^2}+C \end{cases} \quad \text{or} \quad \begin{cases} x=\frac{1}{2}\sinh 2p, \\ y=\frac{2}{3}\cosh^3 p-\cosh p+C \end{cases}$$

—parametric representation of the solution.

444. $y=xC-\frac{1}{4}C^2$ is the general solution; a singular solution is found from the system

$$\left.\begin{aligned} y-xC+\tfrac{1}{4}C^2&=0, \\ \frac{\partial}{\partial C}\left[y-xC+\tfrac{1}{4}C^2\right]&=0, \end{aligned}\right\} \qquad \left.\begin{aligned} y-xC+\tfrac{1}{4}C^2&=0, \\ -x+\frac{C}{2}&=0, \end{aligned}\right\}$$

$$C=2x, \quad y=x^2.$$

We make sure by a check that $y=x^2$ is the solution of Clairaut's equatioa, hence, this is a singular solution (Fig. 88). As is seen from the figure, the parnbola $y=x^2$ is an envelope for the family of straight lines $y=xC-\frac{1}{4}C^2$.

445. $y=Cx+\sqrt{1+C^2}$ is the general solution. For $x=0$, $y=\sqrt{1+C^2}\geqslant 1$. Singular solution: $x^2+y^2=1$. Bearing in mind that the straight lines $y=Cx+$ $+\sqrt{1+C^2}$ intersect the y-axis at points with ordinate $\geqslant 1$, we conclude that the upper half of the circle is a singular solution (Fig. 89). This is an envelope of a family of straight lines.

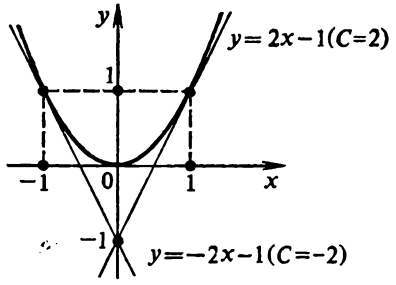

Fig. 88.

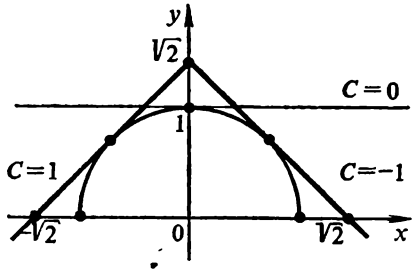

Fig. 89.

446. (a) $y = -\cos x + C_1 x + C_2$; (b) $y = \dfrac{x^4}{24} + C_1 x^2 + C_2 x + C_3$.

447. The given equation does not contain the desired function y in the explicit form. Reduction of the order is obtained by introducing a new function: $z(x) = y'$. We have $z'(x) = y''$; $x^2 z' = z^2$. Thus, we have obtained an equation of the first order with variables separable. It is obvious that $z(x) \equiv 0$ is the solution of the equation, then $y'(x) \equiv 0$, $y(x) = C$ is the solution of the original equation.

Let $z \not\equiv 0$; then, separating the variables, we obtain

$$\frac{dz}{z^2} = \frac{dx}{x^2} \quad (x \neq 0), \qquad \frac{1}{z} = \frac{1}{x} + C_1, \qquad \frac{dx}{dy} = \frac{1}{x} + C_1,$$

$$y = \int \frac{x\,dx}{C_1 x + 1} = \frac{x}{C_1} - \frac{1}{C_1^2} \ln |C_1 x + 1| + C_2 \quad (C_1 \neq 0).$$

If $C_1 = 0$, then $z = x$, $y' = x$, $y = \dfrac{x^2}{2} + C$.

448. $2y = (1 + 2C_1)x^2 + C_1 \cos 2x + C_2 x + C_3$. *Hint.* $z(x) = y''(x)$.

449. $y = -(x + C_1) \ln |C_2(x + C_1)| + x + C_3$; $y = C_1 x + C_2$.

450. The given equation can be solved by the substitution $y'(x) = z(x)$. But it is easily seen that $2yy' = (y^2)'$, therefore the equation can be written in the form

$$(y')' = (y^2)',$$

whence

$$y' = y^2 + p, \qquad x = \int \frac{dy}{y^2 + p}.$$

Considering the cases $p = 0$, $p < 0$, $p > 0$, we obtain:

$$x = -\frac{1}{y} + C_2; \qquad 2c_1 x + C_2 = \ln \left| \frac{y - C_1}{y + C_1} \right| \quad (p = -C_1^2);$$

$$y = C_1 \tan C_1(x - C_2) \quad (p = C_1^2).$$

451. $y \ln |C_1 x| = 1$; $y = C_1 \tan (C_1 \ln |C_2 x|)$; $C_2 x = \left| \dfrac{y - C_1}{y + C_1} \right|^{1/(2C_1)}$.

Hint. Reduce the equation to the form $(xy')' = (y^2)'$.

452. $y^2 = (C_1 + x)^2 + C_2$. *Hint.* $yy'' + y'^2 = (yy')'$.

453. The given equation does not contain the argument x in the explicit form. The order is reduced by introducing a new function: $z(y) = y'(x)$. Hence,

$$y''(x) = z'_y(y) \cdot y'_x = z'_y \cdot z.$$

The equation takes the form

$$2y \frac{dz}{dy} z = z^2 + y^2.$$

This is a homogeneous equation. Solving it, we obtain ($y \not\equiv 0$):

$$z = \pm \sqrt{y^2 + C_1 y}.$$

Further,

$$\frac{dy}{dx} = \pm \sqrt{y^2 + C_1 y}, \qquad \frac{dy}{\sqrt{y^2 + C_1 y}} = \pm dx,$$

$$\pm x + C_2 = \ln \left| y + \frac{C_1}{2} + \sqrt{C_1 y + y^2} \right|.$$

The function $y(x) \equiv 0$ is also a solution.

454. $y \ln |y| + x + C_1 y + C_2 = 0$; $y = C$. *Hint.* $y'(x) = z(x)$.

455. $x = \dfrac{1}{C_1} \ln \left| \dfrac{y}{y + C_1} \right| + C_2$; $y = C$. *Hint.* On introducing a new function $z(y) = y'_x$, we obtain a linear equation with respect to $z(y)$.

456. The given equation contains x and y. But it is a homogeneous second-degree equation with respect to y, y', y''. The order is reduced by introducing a new function $z(x)$ by the formula $y' = yz$ ($y \neq 0$). In this case $y'' = y(z^2 + z')$ and the equation takes the form $xz' = z$. Solving this equation, we obtain $z = C_1 x$. Substituting y'/y for z, we obtain a first-order differential equation: $y' = C_1 xy$. Integrating, we get $y = C_2 \exp(C_1 x^2/2)$. This solution incorporates the solution $y = 0$.

457. $y = C_2 x \exp(-C_1/x)$. *Hint.* $y' = yz(x)$. **458.** $y = C_1 e^{-x} + C_2 e^{3x}$. **459.** $y = C_1 + C_2 e^{2x} + C_3 e^{3x}$.

460. The characteristic equation has the form: $k^4 - 1 = 0$. Its roots can be found by taking the fourth root of unity. But we can also factorize the left-hand member: $(k^2 - 1)(k^2 + 1) = 0$, whence $k_1 = -1$, $k_2 = 1$, $k_3 = -i$, $k_4 = i$. Therefore the general solution of the original equation will be

$$y = C_1 e^{-x} + C_2 e^{x} + C_3 \cos x + C_4 \sin x.$$

461. The roots of the characteristic equation: $k_1 = 1$, $k_2 = k_3 = 2$;

$$y = C_1 e^{x} + e^{2x}(C_2 + C_3 x).$$

462. (a) The characteristic equation $k^4 + 2k^2 + 1 = 0$ is readily reduced to the form $(k^2 + 1)^2 = 0$. Hence, it has the roots $k_1 = k_2 = i$, $k_3 = k_4 = -i$. The general solution can be written as

$$y = e^{ix}(C_1 + C_2 x) + e^{-ix}(C_3 + C_4 x).$$

Taking advantage of Euler's formulas, we have

$$y = (C_1 x + C_2) \cos x + (C_3 x + C_4) \sin x.$$

(b) $y = C_1 e^{x} + C_2 e^{-4x}$.

464. The general solution of a homogeneous equation is already known to us (see Problem 461). It remains to find the particular solution of the nonhomogeneous equation.

(a) The right-hand side of $f(x) = 2e^{3x}$ has a special form (see [3], Sec. 1.16), where $k_0 = 3$ is not a root of the characteristic equation, therefore the partial solution

$$\bar{y} = Ae^{3x},$$

where

$$A = \frac{2}{R_3(3)} = 1, \qquad R_3(k) = k^3 + 5k^2 + 8k - 4.$$

Hence, the general solution of the nonhomogeneous equation has the form

$$y = C_1 e^{x} + e^{2x}(C_2 + C_3 x) + e^{3x}.$$

(b) $k_0 = 1$ is a simple root of the characteristic equation, therefore the partial solution should be sought for in the form $\bar{y} = Axe^{x}$. Determining the derivatives of $\bar{y}$ and substituting them into the equation, we find that $A = 4$. In general, we can prove that

$$A = a/R_n'(k_0),$$

if $f(x) = a \exp(k_0 x)$ and k_0 is a simple root of the characteristic equation. The general solution of the nonhomogeneous equation will be written as

$$y = C_1 e^{x} + e^{2x}(C_2 + C_3 x) + 4xe^{x};$$

(c) $k_0 = 2$ is the root of multiplicity 2 of the characteristic equation. Therefore we seek for a partial solution in the form $\bar{y} = Ax^2e^{2x}$. Determining the required derivatives of $\bar{y}$ and substituting them into the original equation, we find $A = 3/2$. In general, it is possible to prove that if $f(x) = ae^{k_0x}$ and k_0 is the root of multiplicity 2 of the characteristic equation, then

$$A = \frac{a}{R_n''(k_0)}.$$

In this case $R_3''(k) = 6k - 10$, $a = 3$. Thus,

$$y = C_1e^x + e^{2x}(C_2 + C_3x) + \frac{3}{2}x^2e^{2x}.$$

465. See Problem 462. (a) $y = \frac{1}{4}e^x$. (b) Pass to the functions $e^{\pm ix}$ according to Euler's formulas; $k_0 = \pm i$ are the roots of multiplicity 2 of the characteristic equation. $\bar{y} = -\frac{1}{8}x^2 \sin x$; (c) $\bar{y} = -\frac{x^2}{8}(\cos x + \sin x)$; (d) $\bar{y} = \frac{1}{9}\sin 2x$.

466. $y = C_1e^x + C_2e^{-3x} - \frac{\cos x}{10} - \frac{\sin x}{5}$. **467.** (a) $\bar{y} = Ae^x$; (b) $\bar{y} = (Ax^2 + Bx)e^{2x}$; (c) $\bar{y} = A\cos x + B\sin x$; (d) $\bar{y} = (A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5)e^{3x}$; (e) $\bar{y} = A_1x + A_2x^2 + A_3x^3$. Here the right-hand side $P(x)e^{k_0x}$, $k_0 = 0$ is the root of the characteristic equation. (f) $\bar{y} = e^x[(A_1 + A_2x)\sin x + (A_3 + A_4x)\cos x]$.

468. (a) Linearly dependent: $\alpha \cdot 1 + \beta \sin^2 x + \gamma \cos 2x \equiv 0$ for $\alpha = -1$, $\beta = 2$, $\gamma = 1$; (b) linearly dependent; (c) linearly independent, since their Wronskian is not equal to zero; (b) linearly independent ($W[x^2, x^3, x^4] = 2x^6 \not\equiv 0$, $x \not\equiv 0$).

469. (a) $y = C_1x^2 + C_2x^3$. *Hint.* Seek for the partial solutions in the form $y = x^k$. The functions x^2 and x^3 are partial linearly independent solutions of the equation, therefore their linear combination yields the general solution of the equation; (b) $y = C_1x + C_2x^3$; (c) we seek for the solutions in the form $y = x^k$; the characteristic equation $k^2 - 4k + 4 = (k-2)^2 = 0$ has a repeated root $k_1 = k_2 = 2$; $y = x^2$ is a particular solution of the equation. We seek for another solution in the form $y = Ax^2 \ln x$. It is easy to make sure that for any A this is a solution of Euler's equation. Hence, the general solution is $y = (C_1 + C_2 \ln x)x^2$; (d) the equation turns into Euler's equation after multiplication of both sides of the equation by x; $y = C_1 + C_2x^3 + C_3 \ln x$.

471. $y = e^{x^2/2}(C_1 \cos x + C_2 \sin x)$.

472. (a) $y = C_1x + \frac{C_2}{x} + \frac{x^2}{3}$; (b) $y = C_1x + \frac{C_2}{x} + \frac{x^{10}}{99}$.

473. The right-hand sides of the equations have no special form

$$e^{\alpha x}(P_m(x)\cos \beta x + Q_m(x)\sin \beta x).$$

Therefore the partial solution of the nonhomogeneous equation should be found by the method of variation of arbitrary constants.

(a) The general solution of the homogeneous equation has the form

$$y = C_1 \cos x + C_2 \sin x.$$

Regarding $C_1(x)$, $C_2(x)$ as functions of x, let us find them so that the function $\bar{y}(x) = C_1(x)\cos x + C_2(x)\sin x$ be a partial solution of the nonhomogeneous

equation. For this purpose, we have to solve the system (see [3], Sec. 1.17)

$$\begin{cases} C_1'(x)\cos x + C_2'(x)\sin x = 0. \\ -C_1'(x)\sin x + C_2'(x)\cos x = \dfrac{1}{\sin x}. \end{cases}$$

The determinant of this system is the Wronskian $W[\cos x, \sin x] = 1 \neq 0$. Solving the system, we find

$$C_1'(x) = -1, \quad C_2'(x) = \cot x.$$

Integrating, we obtain

$$C_1(x) = -x, \quad C_2(x) = \int \cot x\, dx = \int \frac{d\sin x}{\sin x} = \ln|\sin x|.$$

Hence,

$$\bar{y} = -x\cos x + \sin x \ln|\sin x|,$$

and the general solution of the nonhomogeneous equation will be written as

$$y = C_1\cos x + C_2\sin x - x\cos x + \sin x \ln|\sin x|.$$

We find now the constants C_1 and C_2 from the initial conditions:

$$y(\pi/2) = C_2 = 1, \quad y'(\pi/2) = -C_1 + \pi/2 = 0, \quad C_1 = \pi/2.$$

Thus, the solution of Cauchy's problem will be:

$$y = \frac{\pi}{2}\cos x + \sin x - x\cos x + \sin x \ln|\sin x|.$$

(b) $y = C_1\cos 2x + C_2\sin 2x - x\cos 2x - \dfrac{1}{2}\sin 2x + \sin 2x \ln|\cos x|.$

(c) $y = C_1 + C_2\cos x + C_3\sin x + \ln\left|\tan\left(\dfrac{x}{2} + \dfrac{\pi}{4}\right)\right|$

$$-x\cos x + \sin x \ln|\cos x|.$$

(d) $y = (C_1 + C_2 x)e^{-x} + (-x + x\ln|x|)e^{-x} = (a + bx)e^{-x} + xe^{-x}\ln|x|.$

474. (a) $y = C_1 e^{2x} + C_2 e^{-3x} + \dfrac{2}{3}x^2 + \dfrac{8}{9}x - \dfrac{8}{27},$

$$z = -C_1 e^{2x} + \frac{C_2}{4}e^{-3x} - \frac{x^2}{3} + \frac{2}{9}x - \frac{2}{27}.$$

Hint. Differentiating the first equation, we obtain

$$\ddot{y} + 2\dot{y} + 4\dot{z} = 4. \tag{1}$$

From the first equation we find the function $z = x - \dfrac{1}{4}\dot{y} - \dfrac{1}{2}y$. The derivative $\dot{z}$ is found from the second equation

$$\dot{z} = z - y + x^2 = x + x^2 - \frac{1}{4}\dot{y} - \frac{3}{2}y.$$

Substituting this value of $\dot{z}$ into (1), we obtain a linear non-homogeneous differential equation of the second order with constant coefficients with respect to the function $y(x)$:

$$\ddot{y} + \dot{y} - 6y = 4(1 - x - x^2).$$

(b) The given system can be solved in the same manner as system (a). But we may also apply the theory set forth in Sec. 1.22 of [3]. Let us reduce the system to the form

$$\begin{cases} \left(0+\dfrac{d}{dx}\right)y-z=0, \\ y+\left(0+\dfrac{d}{dx}\right)z=0. \end{cases}$$

This is a homogeneous system, therefore it is reduced to one and the same equation with respect to either y or z. We then write the determinant of the system:

$$D(\lambda)=\begin{vmatrix} 0+\lambda & -1 \\ 1 & 0+\lambda \end{vmatrix}=\lambda^2+1 \qquad \left(\lambda=\frac{d}{dx}\right).$$

The required equation with respect to the function y has the form: $D\left(\dfrac{d}{dx}\right)y=0$ or $\dfrac{d^2y}{dx^2}+y=0$. The general solution of this equation

$$y=C_1\cos x+C_2\sin x.$$

The function z is found from the first equation:

$$z=\frac{dy}{dx}=-C_1\sin x+C_2\cos x.$$

Since the function z satisfies the same differential equation as the function y, we could write at once that

$$z=a\cos x+b\sin x$$

and then would choose the numbers a and b so that the functions y and z satisfy our system (i.e. we substitute the functions y and z into the system and express the constants a and b in terms of C_1 and C_2, or vice versa).

(c) $$D(\lambda)=\begin{vmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{vmatrix}=(\lambda-2)(\lambda+1)^2.$$

The differential equation with respect, say, to the function $x(t)$ has the form

$$\left(\frac{d}{dt}-2\right)\left(\frac{d}{dt}+1\right)^2x(t)=0,$$

and its characteristic equation will be:

$$(k-2)(k+1)^2=0, \qquad k_1=k_2=-1, \qquad k_3=2.$$

The general solution of the differential equation:

$$x(t)=(C_1+C_2t)e^{-t}+C_3e^{2t}.$$

The functions $y(t)$ and $z(t)$ are expressed in a similar way:

$$y(t)=(a+bt)e^{-t}+de^{2t}, \qquad z(t)=(\alpha+\beta t)e^{-t}+\gamma e^{2t}.$$

Substituting these functions into the system, we find that $b=\beta=C_2=0$, $d=\gamma=C_3$, $a+\alpha=-C_1$, where a may be regarded as arbitrary, then $\alpha=-C_1-a$. Thus,

$$x(t)=C_1e^{-t}+C_3e^{2t}, \qquad y(t)=ae^{-t}+C_3e^{2t},$$
$$z(t)=-(C_1+a)e^{-t}+C_3e^{2t};$$

(d) The given system is nonhomogeneous, therefore either of the functions y and z will have its own equation:

$$D\left(\frac{d}{dx}\right)y = \Phi_1(x),$$

where

$$\Phi_1(x) = M_{11}\left(\frac{d}{dx}\right)f_1(x) + M_{21}\left(\frac{d}{dx}\right)f_2(x) = M_{11}\left(\frac{d}{dx}\right)\cdot 1 + M_{21}\left(\frac{d}{dx}\right)x,$$

$M_{sj}\left(\frac{d}{dx}\right)$ is the cofactor of the element b_{sj} of the determinant

$$D(\lambda) = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 + 1 \quad \left(\lambda = \frac{d}{dx}\right),$$

i.e.

$$M_{11}(\lambda) = \lambda, \quad M_{21}(\lambda) = 1, \quad M_{12}(\lambda) = -1, \quad M_{22}(\lambda) = \lambda.$$

Thus,

$$\Phi_1(x) = \frac{d}{dx}\cdot 1 + 1\cdot x = x.$$

Finally, we obtain the following differential equation with respect to the function $y(x)$: $\ddot{y}+y = x$ (the function z satisfies the equation $\ddot{z}+z = 0$). Solving these equations, we get

$$y = C_1\cos x + C_2\sin x + x, \quad z = C_2\cos x - C_1\sin x.$$

475. (a) *Solution.* **Method 1.** Let us write the characteristic equation

$$\begin{vmatrix} 2-\lambda & 1 \\ 3 & 4-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 5 = 0.$$

Its roots are: $\lambda_1 = 1$, $\lambda_2 = 5$. Let us find the eigenvectors $\boldsymbol{\alpha}^1 = (\alpha_1^{(1)}, \alpha_2^{(1)})$ and $\boldsymbol{\alpha}^2 = (\alpha_1^{(2)}, \alpha_2^{(2)})$, corresponding to the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 5$:

$$(2-1)\alpha_1^{(1)} + \alpha_2^{(1)} = 0, \quad \alpha_1^{(1)} + \alpha_2^{(1)} + 0, \quad \alpha_1^{(1)} = -\alpha_2^{(1)},$$

where $\alpha_2^{(1)}$ is an arbitrary number. To make the notation simpler, let us set $\alpha_2^{(1)} = -1$, then $\alpha_1^{(1)} = 1$. And so $\boldsymbol{\alpha}^1 = (1, -1)$. Analogously, form the equation

$$(2-5)\alpha_1^{(2)} + \alpha_2^{(2)} = 0$$

we find $\boldsymbol{\alpha}^2 = (1, 3)$. The solutions of the system will be written in the following way:

$$\mathbf{Y}^1(t) = \{e^t, -e^t\}, \quad \mathbf{Y}^2(t) = \{e^{5t}, 3e^{5t}\}.$$

The general solution:

$$\mathbf{Y}(t) = C_1\mathbf{Y}^1(t) + C_2\mathbf{Y}^2(t) = \{C_1e^t + C_2e^{5t}, \quad -C_1e^t + 3C_2e^{5t}\}.$$

In the expanded form:

$$y_1(t) = C_1e^t + C_2e^{5t}, \quad y_2(t) = -C_1e^t + 3C_2e^{5t}.$$

Method 2. On having found the roots of the characteristic equation $\lambda_1 = 1$, $\lambda_2 = 5$, we can write at once the form of the general solution:

$$y_1(t) = C_1e^t + C_2e^{5t}; \quad y_2(t) = ae^t + be^{5t}.$$

Substituting these functions into our system, we find a and b expressed in terms of C_1 and C_2. Here we escape the process of finding the eigenvectors $\boldsymbol{\alpha}^1$ and $\boldsymbol{\alpha}^2$.

(b) $x(t) = C_1e^{-t}+C_2e^{3t}$, $y(t) = 2C_1e^{-t}-2C_2e^{3t}$.

(c) $y = e^{-x}(C_1\cos x+C_2\sin x)$,

$$z = \frac{1}{5}e^{-x}[(C_2-2C_1)\cos x-(C_1+2C_2)\sin x].$$

(d) $D(\lambda) = -\lambda^3+1 = 0$; $\lambda_1 = 1$, $\lambda_2 = -\frac{1}{2}(1+i\sqrt{3})$, $\lambda_3 = -\frac{1}{2}(1-i\sqrt{3})$—the roots of the characteristic equation:

$$x = C_1e^t+e^{-t/2}\left(C_2\cos\left(\frac{\sqrt{3}}{2}t\right)+C_3\sin\left(\frac{\sqrt{3}}{2}t\right)\right),$$

$$y = C_1e^t+e^{-t/2}\left(\frac{C_3\sqrt{3}-C_2}{2}\cos\left(\frac{\sqrt{3}}{2}t\right)-\frac{C_2\sqrt{3}+C_3}{2}\sin\left(\frac{\sqrt{3}}{2}t\right)\right),$$

$$z = C_1e^t+e^{-t/2}\left(\frac{-C_3\sqrt{3}-C_2}{2}\cos\left(\frac{\sqrt{3}}{2}t\right)+\frac{C_2\sqrt{3}-C_3}{2}\sin\left(\frac{\sqrt{3}}{2}t\right)\right).$$

476. The solution of the homogeneous system is already known to us (see Problem 475, (b)):

$$x = C_1e^{-t}+C_2e^{3t}, \quad y = 2C_1e^{-t}-2C_2e^{3t}.$$

(a) Regarding $C_1(t)$, $C_2(t)$ as functions of t, let us choose them so that the functions

$$\bar{x} = C_1(t)e^{-t}+C_2(t)e^{3t}, \quad \bar{y} = 2C_1(t)e^{-t}-2C_2(t)e^{3t}$$

would be the solutions of the nonhomogeneous system *(Lagrange's method of variation of constants)*. Differentiating these functions and substituting them into the system, we obtain

$$\begin{cases} C_1'(t)e^{-t}+C_2'(t)e^{3t} = 1, \\ 2C_1'(t)e^{-t}-2C_2'(t)e^{3t} = t, \end{cases} \quad C_1'(t) = \frac{2+t}{4}e^t, \quad C_2'(t) = \frac{2-t}{4}e^{-3t}.$$

Integrating, we get

$$C_1(t) = \frac{1}{4}(1+t)e^t, \quad C_2(t) = \frac{1}{36}(-5+3t)e^{-3t};$$

$$\bar{x} = \frac{1}{9}+\frac{t}{3}, \quad \bar{y} = \frac{7}{9}+\frac{t}{3}.$$

The general solution of the nonhomogeneous system has the form:

$$x = C_1e^{-t}+C_2e^{3t}+\frac{1}{9}+\frac{t}{3}, \quad y = 2C_1e^{-t}-2C_2e^{3t}+\frac{7}{9}+\frac{t}{3}.$$

(b) $C_1(t) = \frac{1}{16}(4e^{2t}+e^{4t})$, $C_2(t) = -\frac{1}{4}(t+e^{-2t})$;

$$\bar{x} = \frac{1}{16}(1-4t)e^{3t} \quad \bar{y} = e^t+\frac{1}{8}(1+4t)e^{3t};$$

$$x = C_1e^{-t}+C_2e^{3t}+\frac{1-4t}{16}e^{3t}$$

$$y = 2C_1e^{-t}-2C_2e^{3t}+e^t+\frac{1}{8}(1+4t)e^{3t}.$$

477. $y = 2+x-\frac{x^2}{2}+\frac{5}{6}x^3+\frac{1}{8}x^4+\ldots$. *Hint.* Seek for the solution in the form of the series

$$y = a_0+a_1x+a_2x^2+\ldots,$$

where, by virtue of the initial conditions, $a_0 = 2$, $a_1 = 1$.

479. $y = 2+x-x^2-\frac{x^3}{2}-\frac{x^4}{12}+\ldots$.

480. $y = \frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{6}+\ldots$.

481. (a) The Lyapunov function can be taken in the form $v = x^2+y^2$;

$$\frac{dv}{dt} = \frac{\partial v}{\partial x}\frac{dx}{dt}+\frac{\partial v}{\partial y}\frac{dy}{dt} = -2x^4-2y^4 \leqslant 0.$$

The stationary point is stable.

(b) The Lyapunov function in the form of a quadratic positive definite form does not suit this problem. Let us find it in the form

$$v = x^\alpha+y^\beta$$

with even α and β. We find the total derivative of v (along the solution x, y)

$$\frac{dv}{dt} = \alpha x^{\alpha-1}(2y^3-x^5)+\beta y^{\beta-1}(-x-y^3+y^5).$$

In order that this function be $\leqslant 0$ in the neighbourhood of the origin, it is necessary that the terms of the form $x^{\alpha-1}y^3$ and $y^{\beta-1}x$ be absent. Thus, it must be $\alpha = 2$, $\beta = 4$. In this case

$$\frac{dv}{dt} = -2x^6-4y^6+4y^8 = -2[x^6+2y^6(1-y^2)] \leqslant 0$$

in a sufficiently small neighbourhood of the origin. Besides, $v = x^2+y^4 \geqslant 0$ in the neighbourhood of the origin and $v = 0$ only for $x = y = 0$. Therefore, by Lyapunov's theorem, the solution $x(t) = y(t) \equiv 0$ is stable.

(c) For the function $v = x^2+y^2$

$$\frac{dv}{dt} = 2x(x^3-y)+2y(x+y^3) = 2x^4+2y^4 > 0$$

outside the origin. Hence, by Cetaev's theorem, the zero solution is unstable.

482. (a) $A = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}$, $a_{11}a_{22}-a_{12}^2 = 1 > 0$; the system is elliptic; $a_{11} < 0$, $a_{22} < 0$; the stationary point is a stable node.

(b) $A = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$; $a_{11}a_{22}-a_{12}^2 = 0$, the system is parabolic; $\lambda_1 = a_{11}+a_{22} = 5 > 0$; the stationary point is unstable.

(c) $A = \begin{pmatrix} 1 & 3 \\ 3 & 8 \end{pmatrix}$; $a_{11}a_{22}-a_{12}^2 = -1 < 0$, the system is hyperbolic; the stationary point is unstable.

483. (a) The characteristic equation

$$\begin{vmatrix} 3-\lambda & 0 \\ 2 & 1-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda) = 0$$

has two positive roots $\lambda_1 = 1$, $\lambda_2 = 3$, therefore the stationary point is an unstable node.

(b) The characteristic equation

$$\begin{vmatrix} 1-\lambda & 3 \\ -6 & -5-\lambda \end{vmatrix} = \lambda^2+4\lambda+13 = 0$$

has the roots $\lambda_1 = -2-3i$, $\lambda_2 = -2+3i$. The real part of these roots is negative, therefore the zero solution is a stable focus.

(c) The characteristic equation

$$\begin{vmatrix} 1-\lambda & 0 \\ 2 & -1-\lambda \end{vmatrix} = \lambda^2-1 = 0$$

has two roots $\lambda_1 = -1$, $\lambda_2 = 1$ of opposite signs, hence, the stationary point is a saddle.

(d) The characteristic equation

$$\begin{vmatrix} -2-\lambda & -5 \\ 2 & 2-\lambda \end{vmatrix} = \lambda^2+6 = 0$$

has the complex roots $\lambda_1 = -i\sqrt{6}$, $\lambda_2 = i\sqrt{6}$ with the real part $p = 0$, therefore the stationary point is the centre.

(e) The characteristic equation

$$\begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = 0$$

has a positive multiple root $\lambda_1 = \lambda_2 = 1$, hence, the stationary point is an unstable node.

Chapter 7.

484. $-\infty < x < \infty$; $F(x)$ is continuous and differentiable

$$F'(x) = \frac{1}{x}(\sin 2\pi x - \sin \pi x).$$

485. $\dfrac{b}{2a^2(a^2+b^2)} + \dfrac{1}{2a^3} \operatorname{arc\,tan} \dfrac{b}{a}$. *Hint.* Differentiate the equality with respect to the parameter a.

486. (a) $F'(x) = 2x \exp(-x^5) - \exp(-x^2) - \displaystyle\int_x^{x^2} y^2 \exp(-xy^2)\, dy$;

(b) $F'(x) = \dfrac{2}{x} \ln(1+x^2)$; (c) $F'(x) = -f(x, -x) + 2\displaystyle\int_0^x f'_u(u, v)\, dy$,

where $u = y+x$, $v = y-x$. *Hint.* First make the substitution $z = y-x$ in the integral.

487. (a) $-1/10$; (b) $1/2$. **488.** $\ln(25/24)$. **489.** 50.4. **490.** 12/5.

491. $I = \displaystyle\int_{-1}^{0} dx \int_{-\sqrt{4x+4}}^{\sqrt{4x+4}} f(x, y)\, dy + \int_{0}^{8} dx \int_{-\sqrt{4x+4}}^{2-x} f(x, y)\, dy$ (Fig. 90).

492. $I = \displaystyle\int_0^1 dy \int_{y/2}^{3y} f(x, y)\, dx + \int_1^6 dy \int_{y/2}^{3} f(x, y)\, dx$ (Fig. 91).

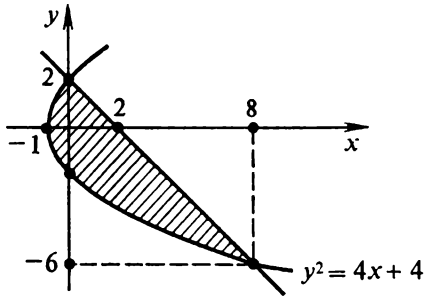

Fig. 90.

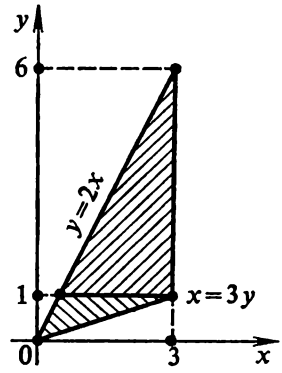

Fig. 91.

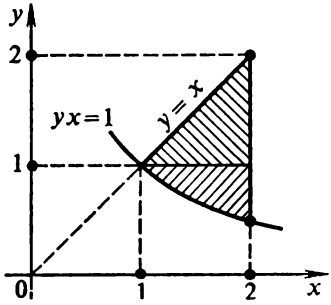

Fig. 92.

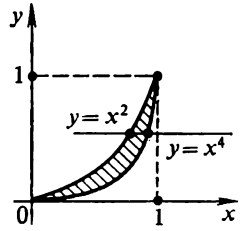

Fig. 93.

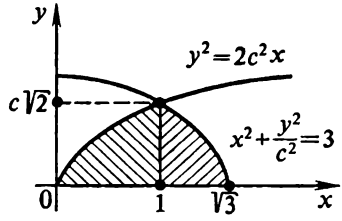

Fig. 94.

493. $I = \int\limits_{1/2}^{1} dy \int\limits_{1/y}^{2} f(x, y)\, dx + \int\limits_{1}^{2} dy \int\limits_{y}^{2} f(x, y)\, dx$ (Fig. 92).

494. $I = \int\limits_{0}^{1} dy \int\limits_{\sqrt{y}}^{\sqrt[4]{y}} f(x, y)\, dx$ (Fig. 93).

495. $I = \int\limits_{0}^{1} dx \int\limits_{0}^{e\sqrt{2x}} f(x, y)\, dy + \int\limits_{1}^{\sqrt{3}} dx \int\limits_{0}^{e\sqrt{3-x^2}} f(x, y)\, dy$ (Fig. 94).

496. $I=\int\limits_0^1 dy\int\limits_y^{2-y}(x+y^2)\,dx=\frac{7}{6}$. **497.** $I=\int\limits_0^1 dy\int\limits_y^{\sqrt{y}} xy^2\,dx=\frac{1}{40}$.

498. $\frac{1}{2}abc\,(a+b+c)$. **499.** 1/48. **500.** 16.

501. (a) $\int\limits_{-a\sqrt{2}}^{a\sqrt{2}} dx\int\limits_{-\sqrt{2a^2-x^2}}^{\sqrt{2a^2-x^2}} dy\int\limits_{\frac{x^2+y^2}{2a}}^{\sqrt{3a^2-x^2-y^2}} f(x,y,z)\,dz$;

(b) $\int\limits_{-\frac{1}{2}R\sqrt{3}}^{\frac{1}{2}R\sqrt{3}} dx\int\limits_{-\sqrt{\frac{3}{4}R^2-x^2}}^{\sqrt{\frac{3}{4}R^2-x^2}} dy\int\limits_{R-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} f(x,y,z)\,dz$.

502. $\frac{1}{3}\pi a^3$.

503. $I=\frac{4}{5}\pi abc$. *Hint.* Since the integrand is even with respect to all the variables, the given integral is equal to eight integrals over the part of ellipsoid found in the first octant. We make the substitution:

$$x=ar\cos t\cos\tau,$$
$$y=br\cos t\sin\tau,\quad \left(0<r\leqslant 1,\ 0\leqslant t\leqslant\frac{\pi}{2},\ 0\leqslant\tau\leqslant\frac{\pi}{2}\right),$$
$$z=cr\sin t$$

The Jacobian of this transformation

$$\left|\frac{D(x,y,z)}{D(r,t,\tau)}\right|=abcr^2\cos t,$$

therefore $I=8abc\int\limits_0^1 r^4\,dr\int\limits_0^{\pi/2} dt\int\limits_0^{\pi/2}\cos t\,dt$.

504. $\pi a^3/6$.

505. $2\pi ab/3$. *Hint.* Take into consideration the evenness of the integrand and introduce the generalized polar coordinates

$$x=ar\cos t,\quad y=br\sin t\quad (0<r\leqslant 1,\ 0<r\leqslant\pi/2).$$

The Jacobian $\frac{D(x,y)}{D(r,t)}=abr$.

506. πR^4.

507. $8a^2/9$. *Hint.* The domain of integration is half of the cylinder of altitude a whose base is a semicircle $(x-1)^2+y^2\leqslant 1$ $(y>0)$. The equation of the semicircle $(x-1)^2+y^2=1$ $(y>0)$ in polar coordinates has the form $\varrho=2\cos\varphi$ $(0\leqslant\varphi\leqslant\pi/2)$ (Fig. 95). Therefore

$$I=\int\limits_0^{\pi/2} d\varphi\int\limits_0^{2\cos\varphi}\varrho^2\,d\varrho\int\limits_0^a z\,dz.$$

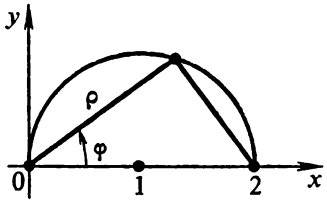

Fig. 95.

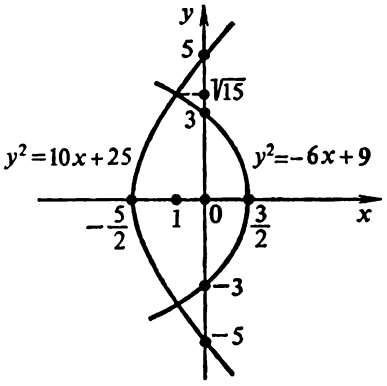

Fig. 96.

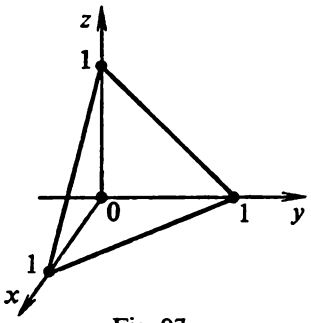

Fig. 97.

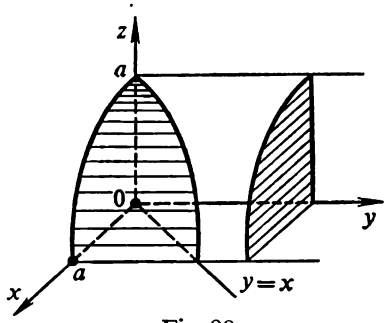

Fig. 98.

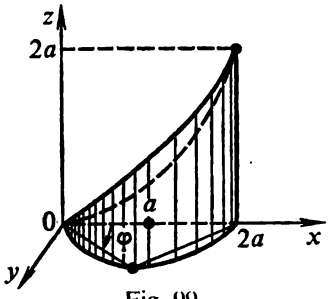

Fig. 99.

508. $4\pi R^5/15$. *Hint.* Pass to polar coordinates.

509. $\frac{16}{3}\sqrt{15}$ (Fig. 96). **510.** $V = \frac{1}{6}$ (Fig. 97).

511. $V = \frac{a^3}{3}$ (Fig. 98).

512. $V = \frac{3}{4}\pi a^3$. *Hint.* To compute the corresponding integral, it is suitable to introduce polar coordinates (Fig. 99). The equation of the semicircle $(x-a)^2+y^2 = a^2$ $(y > 0)$ in polar coordinates will be:

$$\varrho = 2a\cos\varphi \quad (0 < \varphi \leqslant \pi/2).$$

513. $|S| = \iint\limits_D \sqrt{1+(z_x')^2+(z_y')^2}\,dx\,dy$

$$= \iint\limits_D \sqrt{1+\left(\frac{c}{a}\right)^2+\left(\frac{c}{b}\right)^2}\,dx\,dy = \frac{1}{2}\sqrt{a^2b^2+a^2c^2+b^2c^2},$$

where D is a triangle

$$\left\{\begin{array}{l} 0 \leqslant x \leqslant a \\ 0 \leqslant y \leqslant (a-x)\dfrac{b}{a} \end{array}\right\}.$$

514. $|S| = 8a^2 \arcsin(b/a)$. *Hint.* By virtue of symmetry, the sought-for area is equal to the sum of eight areas cut on the spherical surface and found in the first octant.

$$|S| = 8a \iint\limits_D \frac{dx\,dy}{\sqrt{a^2-x^2-y^2}} = 8a \int\limits_0^a dx \int\limits_0^{\frac{b}{a}\sqrt{a^2-x^2}} \frac{dy}{\sqrt{a^2-x^2-y^2}}$$

$$= 8a \int\limits_0^a \arcsin\frac{b}{a}\,dx = 8a^2 \arcsin(b/a)$$

where D is part of an ellipse

$$D = \left\{\frac{x^2}{a^2}+\frac{y^2}{b^2} \leqslant 1, \quad x > 0, \quad y > 0\right\}.$$

515. Let (x_c, y_c) be the centre of gravity. By virtue of symmetry, it is clear that $x_c = 0$. The area D of half of the ellipse equal to $\pi ab/2$ is numerically equal to the mass of the figure, therefore

$$y_c = \frac{2}{\pi ab}\iint\limits_D y\,dx\,dy = \frac{2}{\pi ab}\int\limits_{-a}^{a} dx \int\limits_0^{\frac{b}{a}\sqrt{a^2-x^2}} y\,dy = \frac{4b}{3\pi}.$$

516. (a) $\frac{4}{3}\pi ab^2$. *Hint.* Take advantage of the second Guldin's theorem and Problem 515. (b) $\frac{4}{3}\pi abc$. *Hint.* $V = \iiint\limits_{(V)} dx\,dy\,dz =$

$= 2c \iint\limits_S \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dx\,dy$, where S is an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} \leqslant 1$; then see Problem 505.

517. $x_c = y_c = 0$, $z_c = \frac{2}{5}a$. *Hint.* See Problem 506. Introducing the spherical coordinates

$$\left.\begin{array}{l} x = r\cos\psi\cos\varphi, \\ y = r\cos\psi\sin\varphi, \\ z = r\sin\psi \end{array}\right\} \quad \left(0 \leqslant r \leqslant a, \quad 0 \leqslant \psi \leqslant \frac{\pi}{2}, \quad 0 < \varphi \leqslant 2\pi\right),$$

we obtain

$$z_c = \frac{2}{\pi a^4} \int_0^{2\pi} \int_0^{\pi/2} \int_0^{a} r^4 \cos\varphi \sin\varphi \, d\varphi \, d\varphi \, dr = \frac{4}{5} a \int_0^{\pi/2} \sin\varphi \, d \sin\varphi.$$

518. $x_c = 0$, $y_c = 8/5$. *Hint.* The area of the figure $|S| = 32/3$;

$$y_c = \frac{3}{32} \iint_D y \, dx \, dy = \frac{3}{32} \int_{-2}^{2} dx \int_0^{4-x^2} y \, dy.$$

519. $\frac{512}{15}\pi$. *Hint.* See Problem 518 and the second Guldin's theorem.

521. (a) $\pi/4$; (b) ∞; (c) 1/4.

522. $F(y) = \ln(1+y)$. *Hint.* Integral $F(y)$ converges for any $y > -1$.

$$F'(y) = \int_0^{\infty} e^{-x(y+1)} \, dx = \frac{1}{y+1}$$

converges uniformly for any $y \geqslant y_0 > -1$, therefore differentiation with respect to the parameter y under the integral sign is legitimate in the indicated interval. Taking into account that $F(0) = 0$, we obtain $F(y) = \ln(y+1)$.

523. $F(y) = \sqrt{\pi y}$ $(y > 0)$. *Hint.* As we know,

$$\int_0^{\infty} \exp(-t^2) \, dt = \frac{1}{2}\sqrt{\pi},$$

$$F'(y) = \int_0^{\infty} \exp(-yx^2) \, dx = \frac{1}{\sqrt{y}} \int_0^{\infty} \exp(-t^2) \, dt = \frac{1}{2}\sqrt{\frac{\pi}{y}}.$$

The last integral is uniformly convergent for $y \geqslant y_0 > 0$.

524. Integrating $\int_0^1 x^y \, dx = \frac{1}{y+1}$ with respect to the parameter y between β and α, we obtain

$$\int_\beta^\alpha \left[\int_0^1 x^y \, dx\right] dy = \int_0^1 \left[\int_\beta^\alpha x^y \, dy\right] dx = \int_0^1 \frac{x^y}{\ln x}\Big|_\beta^\alpha dx = \int_0^1 \frac{x^\alpha - x^\beta}{\ln x} \, dx,$$

$$\int_\beta^\alpha \frac{dy}{y+1} = \ln(y+1)\Big|_\beta^\alpha = \ln\frac{\alpha+1}{\beta+1}.$$

Hence,

$$\int_0^1 \frac{x^\alpha - x^\beta}{\ln x} \, dx = \ln\frac{\alpha+1}{\beta+1}.$$

Note that in the domain $0 < x < 1$, $-1 < \beta < y < \alpha$ the original integral converges uniformly.

525. Converges. *Hint.* Consider the circle with the ε-neighbourhood of the origin $U_\varepsilon(0)$ removed and pass to polar coordniates:

$$\iint\limits_{S\setminus U_\varepsilon(0)} \ln\sqrt{x^2+y^2}\,dx\,dy = \int\limits_0^{2\pi}\int\limits_\varepsilon^1 r\ln r\,dr\,d\varphi = 2\pi\int\limits_\varepsilon^1 r\ln r\,dr$$

$$= 2\pi\left(-\frac{\varepsilon^2}{2}\ln\varepsilon-\frac{1}{4}+\frac{\varepsilon^2}{4}\right)\to-\frac{\pi}{2}, \qquad \varepsilon\to 0.$$

526. (a) Converges uniformly by virtue of Weierstrass' test

$$|\cos xy|\leqslant 1, \qquad \left|\int\limits_0^\infty \frac{\cos xy}{1+x^2}\,dx\right| \leqslant \int\limits_0^\infty \frac{dx}{1+x^2} = \arctan x\Big|_0^\infty = \frac{\pi}{2};$$

(b) Converges nonuniformly. The substitution $x\sqrt{y} = u$ shows that

$$\int\limits_0^\infty \sqrt{y}\exp(-yx^2)\,dx = \int\limits_0^\infty \exp(-u^2)\,du,$$

i.e. it is independent of the parameter y. For $y = 0$ the integral is equal to zero. Hence, the integral $F(y)$ is a discontinuous function for $0\leqslant y<\infty$. Consequently, the integral converges nonuniformly. It will be uniformly convergent if $0<y_0\leqslant y<\infty$.

Chapter 8.

527. $\sqrt{5}\ln 2$; the equation of AB: $y = \frac{1}{2}(x-4)$. **529.** 24.

530. $\frac{ab(a^2+ab+b^2)}{3(a+b)}$. *Hint.* The equation of the arc of an ellipse: $y = \frac{b}{a}\sqrt{a^2-x^2}$ $(0\leqslant x\leqslant a)$. Taking x for the parameter, we obtain

$$\int\limits_\Gamma xy\,ds = \int\limits_0^a x\frac{b}{a}\sqrt{a^2-x^2}\sqrt{1+\left(\frac{b}{a}\frac{x}{\sqrt{a^2-x^2}}\right)^2}\,dx$$

$$= \frac{b}{a^2}\int\limits_0^a x\sqrt{a^4+(b^2-a^2)x^2}\,dx.$$

531. $\int\limits_\Gamma y\,ds = 2\int\limits_0^1 \sqrt{x}\sqrt{1+\frac{1}{x}}\,dx = \frac{4}{3}(2\sqrt{2}-1).$

532. Pass to polar coordinates: $x = \varrho\cos\varphi$, $y = \varrho\sin\varphi$. The equation of the circle $x^2+y^2 = 2ax$ or $(x-a)^2+y^2 = a^2$ in polar coordinates has the form $\varrho = 2a\cos\varphi$ $(-\pi/2\leqslant\varphi\leqslant\pi/2)$. The differential of an arc of this circle will be

$$ds = \sqrt{f'(\varphi)^2+f(\varphi)^2}\,d\varphi = \sqrt{4a^2\sin^2\varphi+4a^2\cos^2\varphi}\,d\varphi = 2a\,d\varphi.$$

Therefore

$$\int_{\Gamma}(x-y)\,ds=\int_{-\pi/2}^{\pi/2}(2a\cos^2\varphi-2a\cos\varphi\sin\varphi)\,2a\,d\varphi$$

$$=4a^2\int_{-\pi/2}^{\pi/2}\cos^2\varphi\,d\varphi=2a^2\pi.$$

533. $m=\dfrac{b^2}{2}+\dfrac{a^2b}{2\sqrt{a^2-b^2}}\arcsin\dfrac{\sqrt{a^2-b^2}}{a}$ $(a>b)$.

535. $2\pi^2b\sqrt{a^2+b^2}$.

536. $x_c=\dfrac{4}{3}a$, $y_c=\dfrac{4}{3}a$. *Hint.* The length of the arc of the half-arch of a cycloid

$$|\Gamma|=\int_0^{\pi}\sqrt{x'(t)^2+y'(t)^2}\,dt=4a;$$

$$x_c=\frac{1}{|\Gamma|}\int_0^{\pi}x\sqrt{x'(t)^2+y'(t)^2}\,dt=\frac{4}{3}a,$$

$$y_c=\frac{1}{|\Gamma|}\int_0^{\pi}y\sqrt{x'(t)^2+y'(t)^2}\,dt=\frac{4}{3}a.$$

537. $2\pi a^2\sqrt{a^2+b^2}$. *Hint.*

$$I^{(z)}_{x=0,\,y=0}=\int_{\Gamma}(x^2+y^2)\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}\,dt.$$

539. 2. *Hint.* Take advantage of the property: $\int_{\Gamma}=\int_{\Gamma_1}+\int_{\Gamma_2}$, $\Gamma=\Gamma_1+\Gamma_2$.

540. (a) 512/15; (b) 64/3; (c) 0. **541.** (a) 1; (b) 1; (c) 1.
542. $\operatorname{grad}u=\{2x+y,\ 4y+x,\ 6z-6\}$. **543.** (a) $z^2=xy$; (b) $x=y=z$.
544. $\operatorname{curl}\mathbf{a}=\{0,0,0\}$. **545.** $\operatorname{curl}\mathbf{a}=\{1,1,0\}$.
546. (a) Has, since $\operatorname{curl}\mathbf{a}=\mathbf{o}$ and R^3 is a simply connected domain; (b) has, $\operatorname{curl}\mathbf{a}=\mathbf{o}$; (c) does not have, $\operatorname{curl}\mathbf{a}=\{0,1-y,z\}\neq\mathbf{o}$ in R^3.
548. (a) and (b) are exact differential equations; (c) is not an exact differential equation $\left(\operatorname{curl}\{2-y,x\}\neq\mathbf{o}\text{ or }\dfrac{\partial}{\partial y}(2-y)\neq\dfrac{\partial x}{\partial x}\right)$.

549. $x^2+x^3y-y^3=C$. **550.** $3x^2y-y^3=C$. **551.** $xe^{-y}-y^2=C$.

552. $x+\dfrac{x^3}{y^2}+\dfrac{5}{y}=C$. *Hint.* The potential function $U(x,y)$ for the vector $\mathbf{a}=\left\{\dfrac{3x^2+y^2}{y^2},\ -\dfrac{2x^3+5y}{y^3}\right\}$ should be found as the integral of the second kind of the vector $\mathbf{a}$. Any curve not intersecting the x-axis may be taken for the

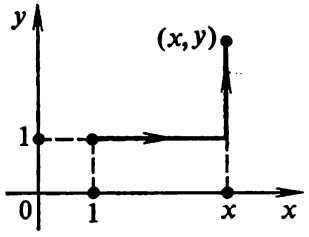

Fig. 100.

integration path. For instance, we may take the polygonal line connecting the points (1, 1), $(x, 1)$, (x, y) $(x > 0, y > 0)$ (see Fig. 100). If the point (x, y) lies in the lower half-plane, then for the initial point we take any point lying below the x-axis, for instance, the point $(0, -1)$.

553. $\operatorname{curl}\{yz, xz, xy\} = \mathbf{o}$; $U(x, y, z) = xyz$. The general solution: $xyz = C$. Any of the variables can be regarded as a function of the two other independent variables.

554. $xy+xz+yz = C$.

555. $y(x^2+z^2) = C$. **556.** $\iint\limits_{\Omega} (x^2+y^2)\,dx\,dy$.

557. $\iint\limits_{\Omega} e^{xy}(y-x)\,dx\,dy$. **558.** 0. **559.** $-1/3$.

560. $\pi R^4/2$. *Hint.* Apply Green's formula and pass to polar coordinates.

561.
$$m\Omega = \frac{1}{2}\int\limits_{\Gamma} (-y\,dx+x\,dy)$$
$$= \frac{1}{2}\int\limits_{0}^{2\pi} (a\sin^3 t\cdot 3a\cos^2 t\sin t + a\cos^3 t\cdot 3a\sin^2 t\cos t)\,dt = \frac{3}{8}\pi a^2.$$

562. $\operatorname{curl}\mathbf{a} = \mathbf{o}$. This means that in the plane, vector $\mathbf{a}$ has a potential function. The work done by vector $\mathbf{a}$ is a line integral of the second kind which is independent of the path of integration; hence, the work done is also independent of the shape of the path of displacement:

$$A = \int\limits_{\Gamma} (\mathbf{a}\,d\mathbf{s}) = \int\limits_{\Gamma} (2xy\,dx+x^2\,dy),$$

where Γ is an arbitrary curve connecting the points (1, 1) and (2, 5). Computing the integral along a concrete curve, we just obtain the numerical value of the work done. Here it is better to find the potential function by solving the exact differential equation: $2xy\,dx+x^2\,dy = 0$ which is an equation with the variables separable. Solving, we obtain $U(x, y) = x^2y$. Now the work $A = U(2, 5)-U(1, 1) = 20-1 = 19$.

564. $\frac{8}{3}\pi a^4$.

565. $m = \int\limits_{S} \sqrt{x^2+y^2}\,dS = \iint\limits_{\Omega} \sqrt{x^2+y^2}\sqrt{1+z_x^2+z_y^2}\,dx\,dy$, where Ω is the part of a circle $x^2+y^2 \leqslant a^2$, found in the first quadrant: $z = \sqrt{a^2-x^2-y^2}$, $m = \frac{1}{8}\pi^2 a^3$.

566. $\pi^2[a\sqrt{1+a^2}+\ln(a+\sqrt{1+a^2})]$. *Hint.* The element of the area of the helicoid: $dS = |\mathbf{r}_u|\cdot|\mathbf{r}_v|\,du\,dv = \sqrt{1+u^2}\,du\,dv$.

568. $\pi a^4/2$. *Hint.* Reduce each term to a double integral along the corresponding projection on the coordinate planes.

569. 0. *Hint.* Carry out the computation for each of the four faces separately. For instance, on the lower face S_1^* (an oriented triangle) the outward normal $\mathbf{n}(A) = -\mathbf{k}$, $z = 0$. Therefore

$$\int_{S_1^*} (yz\,dy\,dz + xz\,dx\,dz + xy\,dx\,dy) = \int_{S_1^*} xy\,dx\,dy = -\int_{\Delta_1} xy\,dx\,dy = -\frac{a^4}{24},$$

where

$$\Delta_1 = \begin{Bmatrix} 0 \leqslant x \leqslant a \\ 0 \leqslant y \leqslant a-x \end{Bmatrix}.$$

Analogously, we proceed for the other faces lying in the coordinate planes. For the face S_4^* lying in the plane $x+y+z=a$ the cosine of the angle made by the outward normal with the z-axis is determined by the equality

$$\cos(\mathbf{n}, z) = 1/\sqrt{1+z_x^2+z_y^2},$$

therefore

$$\int_{S_4^*} (yz\,dy\,dz + xz\,dx\,dz + xy\,dx\,dy)$$

$$= \iint_{\Delta_3} yz\,dy\,dz + \iint_{\Delta_2} xz\,dx\,dz + \iint_{\Delta_1} xy\,dx\,dy = \frac{1}{8}a^4.$$

Finally,

$$\int_{S^*} \ldots = \int_{S_1^*} \ldots + \int_{S_2^*} \ldots + \int_{S_3^*} \ldots + \int_{S_4^*} \ldots = -\frac{a^4}{8} + \frac{a^4}{8} = 0.$$

570. $\operatorname{div} \mathbf{a} = 3(x^2+y^2+z^2)$. **571.** $\operatorname{div} \mathbf{a} = \dfrac{2}{r} f(r) + f'(r)$.

572. $\operatorname{div}(\operatorname{grad} u) = 6$.

573. (a) $\operatorname{curl} \mathbf{a} = \mathbf{o}$;

(b) $$\operatorname{curl}(f(r)\cdot\mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f(r)c_1 & f(r)c_2 & f(r)c_3 \end{vmatrix} = \frac{f'(r)}{r}\,\mathbf{c}\times\mathbf{r}.$$

574. (a) 0. Vector $\mathbf{a} = \{yz, zx, xy\}$, therefore $\operatorname{div} \mathbf{a} = 0$.

(b) $2\iiint_Q (x+y+z)\,dx\,dy\,dz$; here $\mathbf{a} = \{x^2, y^2, z^2\}$.

(c) $2\iiint_Q \dfrac{dx\,dy\,dz}{\sqrt{x^2+y^2+z^2}}$; here $\mathbf{a} = \left\{\dfrac{x}{r}, \dfrac{y}{r}, \dfrac{z}{r}\right\}$, $r = \sqrt{x^2+y^2+z^2}$.

(d) $\iiint_G \Delta u\, dx\, dy\, dz$; here $\mathbf{a} = \operatorname{grad} u$; $\operatorname{div}(\operatorname{grad} u) = \Delta u$, where $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$.

575. 1/2; $\operatorname{div} \mathbf{a} = 3$; $\iiint_G 3\, dx\, dy\, dz = 3 \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz$.

576. 0 (see Problem 567). **577.** 0.

579. -4π. *Hint.* $\mathbf{a} = \{y-z, z-x, x-y\}$, $\mathbf{n} = \{1/\sqrt{2}, 0, 1/\sqrt{2}\}$,

$$\int_\Gamma ((y-z)\,dx + (z-x)\,dy + (x-y)\,dz) = \int_S -\frac{4}{\sqrt{2}}\,dS,$$

where S is an ellipse lying in the plane $x+z=1$. As is seen from Fig. 33, its semimajor axis is equal to $\sqrt{2}$, and the semiminor axis to 1. The area of this ellipse is $\pi\sqrt{2}$. Prior to performing necessary computations, write the equation of the ellipse in the parametric form taking z ($0 \leqslant z \leqslant 2$) for the parameter: $x = 1-z$, $y = \pm\sqrt{2z-z^2}$. We then divide the integral over Γ into two parts: for $y > 0$ and $y < 0$. Note that in the first case z varies from 0 to 2 and in the second from 2 to 0.

580. 0.

Chapter 9.

583. $\forall x$, if $\alpha > 1$; $0 < x < 2\pi$, if $0 < \alpha \leqslant 1$. *Hint.* For $0 < \alpha \leqslant 1$ apply Dirichlet's test ($\alpha_k = k^{-\alpha}$, $\beta_k(x) = \cos kx$).

584. As many as desired. **585.** $-\infty < x < \infty$.

586. (a) $2\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n}$. (b) $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$.

(c) $\frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\cos(2k+1)x}{(2k+1)^2}$.

(d) $\frac{\pi}{4} - \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{\cos(2k+1)x}{(2k+1)^2} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sin kx}{k}$.

(e) $-\frac{\pi}{4} + \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)x}{(2n+1)^2} + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n}$.

(f) If a is an integer, then $\sin ax$; if a is not an integer, then

$$\frac{2\sin a\pi}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{n \sin nx}{a^2 - n^2}.$$

(g) $\frac{2}{\pi} \sinh a\pi \left[\frac{1}{2a} + \sum_{n=1}^{\infty} (-1)^n \frac{a\cos nx}{a^2+n^2}\right]$.

588. (a) $2\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n}$, $\frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\cos(2k+1)x}{(2k+1)^2}$

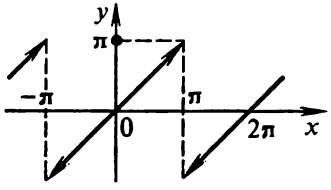

Fig. 101.

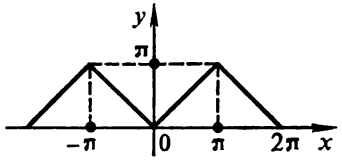

Fig. 102.

Hint. When expanding the function into a series in terms of sines, we extend the function $f(x)$ to $(-\pi, 0)$ in an odd way, and then periodically over the entire number axis (Fig. 101). Further, see Problem 586, (a). When expanding into a series in terms of cosines we extend the function $f(x)$ onto $(-\pi, 0)$ in an even manner (Fig. 102). Then see Problem 586, (c). Note that in this case the extended periodical function is continuous on the entire axis. The Fourier series converges uniformly throughout the axis.

(b) $\sum\limits_{n=1}^{\infty} \dfrac{\sin 2nx}{2n}$ (Fig. 103); $\quad \dfrac{2}{\pi} \sum\limits_{n=0}^{\infty} \dfrac{\cos(2n+1)x}{(2n+1)^2}$ (Fig. 104).

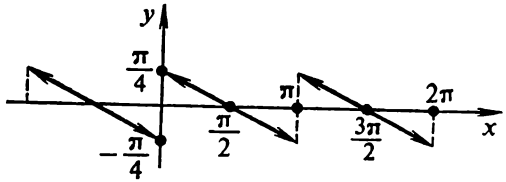

Fig. 103.

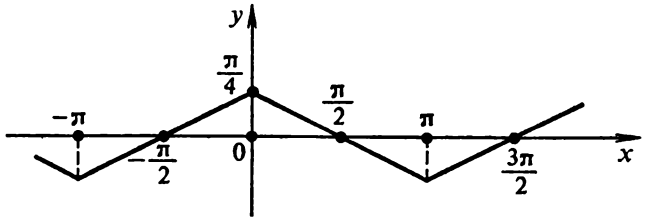

Fig. 104.

589. $\dfrac{l}{2} - \dfrac{4l}{\pi^2} \sum\limits_{n=0}^{\infty} (2n+1)^{-2} \cos\left(\dfrac{2n+1}{l}\pi x\right)$.

590. (a) π; (b) $\pi/4$.

591. (a) $\|f\| = 1/\sqrt{5}$; (b) $\|f\| = \sqrt{\pi/2}$; (c) $\|f\| = \sqrt{(e^2-1)/2}$; (d) $\|f\| = 1/\sqrt{3}$.

592. $0 \leqslant \alpha < 1/2$.

593. For any $\alpha > 0$ the sequence converges to zero nonuniformly $\left(\max\limits_{0 \leqslant x \leqslant \pi} f_n(x) = 1\right)$. For $\forall \alpha > 0$ converges to zero in the mean.

595. $F(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos ts\, dt = \sqrt{\frac{2}{\pi}} \int_0^a \cos ts\, dt = \sqrt{\frac{2}{\pi}} \frac{\sin as}{s}$.

596. $f(x) = \begin{cases} 1, & 0 < x < a, \\ 0, & x > a. \end{cases}$ *Hint.* Take advantage of the repeated Fourier integral

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos xs\, ds \int_0^\infty f(t) \cos ts\, dt,$$

valid for the function $f(x)$ which is piecewise continuous and absolutely integrable on $(0, \infty)$ (see Problem 595).

597. $F(s) = \frac{1}{\sqrt{2\pi}} \left[\frac{\sin (1+s)\, a}{1+s} + \frac{\sin (s-1)\, a}{s-1} \right]$.

598. $Q(s, r) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-sx} \frac{\sin rx}{x}\, dx = \sqrt{\frac{2}{\pi}} \arctan \frac{r}{s}$. *Hint.* Differentiating with respect to the parameter r, we obtain ([3], Sec. 4.14, (4)):

$$\frac{\partial Q}{\partial r} = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-sx} \cos rx\, dx = \sqrt{\frac{2}{\pi}} \frac{s}{s^2+r^2}.$$

Hence,

$$Q = \sqrt{\frac{2}{\pi}} \arctan \frac{r}{s} + C, \quad Q(s, 0) = 0, \quad C = 0.$$

599. (a) $\frac{\pi}{2a^2}(1 - e^{-as})$ (see Problem 598 and [3], Sec. 4.14, (6)); (b) 63/697; (c) 51/290.

Chapter 10.

600. $A \equiv \left(\int_{-\pi}^{\pi} |u(\varrho, \theta) - f(\theta)|^2\, d\theta \right)^{1/2} \leqslant \frac{\pi}{\sqrt{2}} (1 - \varrho) \to 0$ for $\varrho \to 1 - 0$.

Hint. The Fourier series of the function $f(\theta)$ has the form (see Problem 586, (c)):

$$f(\theta) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\cos (2k+1)\, \theta}{(2k+1)^2}.$$

The harmonic function (in the unit circle) generated by the function $f(\theta)$ can be written in the form ([3], Sec. 5.3):

$$u(\varrho, \theta) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \varrho^{2k+1} \frac{\cos (2k+1)\, \theta}{(2k+1)^2}.$$

Hence, on the basis of Parseval's equality, we obtain

$$A = \left(4\sum_0^\infty (1-\varrho^{2k+1})^2 \frac{1}{(2k+1)^4}\right)^{1/2}$$

$$= \left(4\sum_0^\infty (1-\varrho)^2(1+\varrho+\ldots+\varrho^{2k})^2 \frac{1}{(2k+1)^4}\right)^{1/2}$$

$$\leqslant (1-\varrho)\left(4\sum_0^\infty \frac{1}{(2k+1)^2}\right)^{1/2} = (1-\varrho)\left(4\cdot\frac{\pi^2}{8}\right)^{1/2} = \frac{\pi}{\sqrt{2}}(1-\varrho)$$

(see Problem 587).

Remark. The harmonic function $u(\varrho, \theta)$, tends to $f(\theta)$ along each radius (Fig. 105) as $\varrho \to 1-0$, also in the ordinary sense, but the rate of convergence will be somewhat worse than $(1-\varrho)$.

Indeed, for a fixed $\varrho(1/2 < \varrho < 1)$ let us choose a natural number N so that $1/(N+1) < 1-\varrho \leqslant 1/N$, then (see Problem 376)

$$|u(\varrho, \theta)-f(\theta)|$$

$$\leqslant \frac{4}{\pi}\sum_0^\infty (1-\varrho^{2k+1})\frac{|\cos(2k+1)\theta|}{(2k+1)^2}$$

$$\leqslant c\sum_0^N \frac{1-\varrho^{2k+1}}{(2k+1)^2} + c\sum_{N+1}^\infty \frac{1}{(2k+1)^2}$$

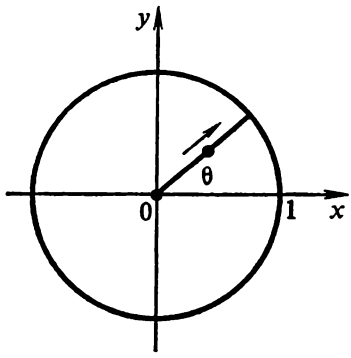

Fig. 105.

$$\leqslant c(1-\varrho)\sum_0^N \frac{1}{2k+1} + \frac{c}{N+1} \leqslant c(1-\varrho)\ln N + \frac{c}{N+1}$$

$$\leqslant c(1-\varrho)\ln\frac{1}{1-\varrho} + c(1-\varrho) \leqslant c(1-\varrho)\ln\frac{1}{1-\varrho} \to 0, \quad \varrho \to 1-0,$$

where the constants c are, generally speaking, distinct in different inequalities.

601. Let us expand the function $f(x)$ into a series in terms of sines (see Problem 588, (b))

$$f(x) = \sum_1^\infty \frac{\sin 2nx}{2n}.$$

The solution of the problem under consideration has the form (see [3], Sec. 4.7, (11)):

$$u(x, t) = \sum_1^\infty \frac{1}{2n}\exp(-4n^2t)\sin 2nx.$$

Using Parseval's equality, we have

$$A = \left[\sum_1^\infty \frac{\pi}{4n^2}(1-\exp(-4n^2t))^2\right]^{1/2}.$$

Let us fix t $(0 < t < 1)$ and choose a natural number N such that $1/(N+1)^2 \leqslant t < 1/N^2$. Then

$$A \leqslant \left(\sum_{n=1}^N \frac{\pi}{4n^2}(1-\exp(-4n^2t))^2 + \sum_{N+1}^\infty \frac{1}{4n^2}\right)^{1/2}.$$

Applying Lagrange's theorem and using the estimates from Problem 376, we obtain

$$A \leqslant \left(\sum_1^N \frac{\pi}{4n^2}(4n^2 t \exp(-4n^2\xi))^2 + \frac{c}{N+1}\right)^{1/2}$$

$$\leqslant c\left(\sum_1^N t^2 n^2 + \frac{1}{N+1}\right)^{1/2} \leqslant c\left(t^2 N^3 + \frac{1}{N+1}\right)^{1/2} \leqslant ct^{1/4} \to 0, \quad t \to 0.$$

602.

$$u(x, t) = \frac{4}{\pi}\sum_{n=0}^{\infty}(-1)^n\left[\frac{\cos(2n+1)t}{(2n+1)^2} + \frac{\sin(2n+1)t}{(2n+1)^3}\right]\sin(2n+1)x$$

(see [3], Sec. 5.5, (8), (11)).

603. Using D'Alembert's formula, we have

$$u(x, t) = \frac{1}{2}\left[\frac{1}{1+(x-t)^2} + \frac{1}{1+(x+t)^2}\right] + \frac{1}{2}[\arctan(x+t) - \arctan(x-t)].$$

605.

$$u(x, y) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{yf(t)\,dt}{(x-t)^2+y^2} = \frac{y}{\pi}\int_{-l}^{l}\frac{dt}{(x-t)^2+y^2}$$

$$= \frac{y}{\pi}\int_{-l-x}^{l-x}\frac{d\xi}{\xi^2+y^2} = \frac{1}{\pi}\left[\arctan\frac{l-x}{y} + \arctan\frac{l+x}{y}\right].$$

If $u = 1/2$, then $\arctan((l-x)/y) + \arctan((l+x)/y) = \pi/2$, i.e. $(l^2-x^2)/y^2 = 1$—a semicircle ($y > 0$) of radius l with centre at the origin.

606.

$$u(x, t) = \frac{1}{\sqrt{1+4t}}\exp\left(-\frac{x^2}{1+4t}\right).$$

Chapter 11.

608. (a) $|z| = 5$; (b) $|z| = 1$; (c) $|z| = 4$.

609. (a) $z = 2[\cos(-2\pi/3) + i\sin(-2\pi/3)]$; (b) $2(\cos(3\pi/4) + i\sin(3\pi/4))$.

610. (a) $2e^{i\pi}$; (b) $1 \cdot e^{i\pi/2}$; (c) $2e^{-2\pi i/3}$; (d) $1 \cdot \exp\left(\alpha - \frac{\pi}{2}\right)i$.

612. (a) 1728; (b) 1.

613. (a) $\sqrt[8]{2}\exp\left(\frac{-\pi+8k\pi}{16}i\right)$ $(k = 0, 1, 2, 3)$; (b) $\pm 1, \pm i$.

614. $\bar{w} = x - iy^2$, $\frac{1}{w} = \frac{\bar{w}}{w\bar{w}} = \frac{x-iy^2}{x^2+y^4}$.

616. (a) A circle of radius r centred at the point z_0; (b) ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $b^2 = a^2 - c^2$. *Hint.* $|z-c|$ is the distance from the point $(c, 0)$ to $z = (x, y)$. By definition, ellipse is the locus of a point which moves in a plane so that the sum of its distances from two fixed points in that plane is constant.

(c) $\operatorname{Re}\left(\frac{1}{w}\right) = \frac{x}{x^2+4y^2} = \frac{1}{2}$, $\operatorname{Im}\left(\frac{1}{w}\right) = \frac{-2y}{x^2+4y^2} = \frac{1}{4}$. These are ellipses: $(x-1)^2 + 4y^2 = 1$, $\frac{x^2}{4} + (y+1)^2 = 1$;

(d) $xy = 1$—a hyperbola; (e) $y = x^2$—a parabola.

617. (a) -1; (b) $-(5+12i)/13$.

619. (a) A circle: $|w| = 2$ or $u^2+v^2 = 4$. For $z = \varrho e^{i\varphi}$ $w = \frac{1}{\varrho} e^{-i\varphi}$, and consequently, the circle $|w| = 2$ is traced clockwise if z traverses the circle $|z| = 1/2$ anticlockwise.

(b) The ray coming along the bisector of the fourth quadrant from ∞ into 0.

(c) The axis Ov except the point $O = (0, 0)$. If the point z moves along the y-axis from $-\infty$ to $+\infty$, then the point w moves first from 0 to $+\infty$, and then from $-\infty$ to 0.

620. $\lim\limits_{z \to i} \bar{z} = -i$.

621. Does not exist, since $\operatorname{Re} w = \frac{x^2-y^2}{x^2+y^2}$ has no limit as $x \to 0$, $y \to 0$.

622. Yes.

623. (a) $w' = 2z$; (b) the derivative does not exist at the points belonging to the z-plane; (c) has no derivative; (d) has the derivative equal to zero only at the point 0.

624. $f'(z) = \cos z$, $|f'(z)|_{z=0} = |\cos 0| = 1$, $|\arg f'(z)|_{z=0} = |\arg \cos 0| = 0$.

625. (a) $z \neq 0$; $w' = 3z^2$; (b) at the points z which are not zeros of the function $\sin z$, i.e. $z \neq (k\pi, 0)$ $(k = 0, \pm 1, \pm 2, \ldots)$; (c) $z \neq -1$.

626. All the given functions are analytic except (b).

627. (a) $f(z) = z^2+2z+Ci$, where $\operatorname{Im} C = 0$; (b) $f(z) = z^2(2-i)/2+Ci$, where $\operatorname{Im} C = 0$; (c) $f(z) = u+iv = (C-z \ln z)\, i$, where $v = r\varphi \sin \varphi - r \ln r \cos \varphi + C$, $\operatorname{Im} C = 0$ and $\ln z = \ln r + i\varphi$.

629. $u = C \operatorname{arc\,tan} \frac{y}{x} + C_1$.

630. $\frac{w-w_1}{w-w_2} : \frac{w_3-w_1}{w_3-w_2} = \frac{z-z_1}{z-z_2} : \frac{z_3-z_1}{z_3-z_2}$, where z_1, z_2, z_3 and w_1, w_2, w_3 are arbitrary triples of real numbers coming in an increasing order.

632. $w = z^4$. **633.** (a) $w = e^z$; (b) $w = e^{2z}$. **635.** $w = (1+i)(1-z)$.

636. $w = \frac{2z-5}{10-z}$.

637. The integrand can be written in the form:

$1+i-2\bar{z} = (1-2x)+i(1+2y)$. (a) $2(i-1)$; (b) $-2+\frac{4}{3}i$; (c) -2.

638. 2.

639. $1+\cosh \pi - \pi(1+i)\sinh \pi$. *Hint.* Since the function $w = z \cosh z$ is analytic throughout the plane, we may integrate along path connecting the points 0 and $\pi(1+i)$. Take advantage of the formulas: $\cosh iz = \cos z$, $\sinh iz = i \sin z$.

640. (a) $7+19i$; (b) $-i/e$. **641.** (a) $I = 0$, since the integrand is analytic in the circle $|z-2| \leqslant 1$; (b) $I = -\frac{\pi}{3} i$; (c) $I = \frac{\pi i}{3} e^{36}$.

642. (a) $2\pi i$; (b) $2\pi i(-e^{-1})$. *Hint.* $\int\limits_C \ldots = \int\limits_{C_1} \ldots + \int\limits_{C_2} \ldots$, where C_1 and C_2 are the contours described about the singular points $z = -1$ and $z = 0$; C_1 and C_2 lie entirely inside the contour C and do not intersect each other.

644. $-\frac{\pi i}{2e}$. **645.** ∞. **646.** 0. **647.** ∞. **648.** ∞. **649.** 1. **650.** 1. **651.** 1.

652. $\sum_0^\infty (-1)^n (n+1) z^n$ ($|z| < 1$). *Hint.* Differentiate the expansion

$$\frac{1}{1+z} = \sum_0^\infty (-1)^n z^n \quad (|z| < 1).$$

653. $\frac{1}{3} \sum_0^\infty [(-1)^{n+1} - 2^{-n-1}] z^n$ ($|z| < 1$). *Hint.* First expand the function in partial fractions.

655. $-\frac{1}{3} + \frac{2}{3^2}(z-3) - \frac{2^2}{3^3}(z-3)^2 + \ldots \left(|z-3| < \frac{3}{2}\right)$.

656. (a) $|z| > 2$; (b) $|z+1| > 1/4$.

657. (a) $-\frac{1}{2} \sum_0^\infty \left(\frac{z}{2}\right)^n + \sum_0^\infty z^n = \sum_0^\infty (1 - 2^{-n-1}) z^n$;

(b) $-\sum_0^\infty \frac{z^n}{2^n} - \sum_1^\infty \frac{1}{z^n}$; (c) $\sum_0^\infty (2^n - 1) z^{-n-1}$.

658. $\frac{1}{4(z-1)^2} - \frac{1}{4(z-1)} + \frac{3}{16} - \frac{1}{8}(z-1) + \frac{5}{64}(z-1)^2 - \ldots$

$$= \sum_{n=-2}^\infty (-1)^n \frac{n+3}{2^{n+4}} (z-1)^n \quad (0 < |z-1| < 2).$$

Hint. It is possible to take advantage of the formula for the coefficients of the Laurent series

$$c_n = \frac{1}{2\pi i} \int_\gamma f(z) \frac{dz}{(z-1)^{n+1}},$$

where γ is an arbitrary circle centred at the point $z_0 = 1$ and lying in the annulus $0 < |z-1| < 2$. The function $f(z) = 1/(z^2-1)^2$ is analytic in this annulus.

It is also possible to use the method of expanding $f(z)$ in partial fractions:

$$f(z) = \frac{1}{(z^2-1)^2} = \frac{1}{4}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)^2$$

$$= \frac{1}{4(z-1)^2} - \frac{1}{4(z-1)} + \frac{1}{4(z+1)} + \frac{1}{4(z+1)^2}.$$

When expanding the functions $\frac{1}{z+1}$ and $\frac{1}{(z+1)^2}$ in powers of $(z-1)$, use the technique applied in Problem 654.

659. $\frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - \frac{z^5}{7!} + \ldots$ ($|z| > 0$).

660. $z^3 + z^2 + \frac{z}{2!} + \frac{1}{3!} + \frac{1}{4!\,z} + \ldots$ ($|z| > 0$).

661. (a) Removable singularity; (b) pole of the fourth order; (c) essential singularity.

662. (a) $z = 0$—a removable singularity, $z = \infty$—an essential singularity; (b) $z = 0$—an essential singularity, $z = \infty$—a removable singularity; (c) $z = 0$—an essential singularity, $z = \infty$—a removable singularity; (d) $z_n = \left(n + \frac{1}{2}\right)\pi i$—simple poles ($n = 0, \pm 1, \pm 2, \ldots$). *Hint.* The numbers z_n are zeros of the function $w = \cosh z$. In the neighbourhood of z_n the function $\tanh z$ can be represented in the form

$$\tanh z = \frac{\cosh (z - z_n)}{(z - z_n)\dfrac{\sinh (z - z_n)}{z - z_n}}.$$

$z = \infty$ is an essential singularity, since the limit of $\tanh z$, as $z \to \infty$, does not exist: for $z = x$ $\tanh x \to 1$, $x \to \infty$; for $z = iy$ $\tanh z = \dfrac{\sinh iy}{\cosh iy} = \dfrac{i \sin y}{\cos y} =$ $= i \tan y$ and for $y \to \infty$ the limit does not exist.

665. (a) $\operatorname{Res}_{z=0} f(z) = 0$, $\operatorname{Res}_{z=\infty} f(z) = 0$.

(b) $\operatorname{Res}_{z=0} f(z) = \dfrac{1}{3!}$, $\operatorname{Res}_{z=\infty} f(z) = -\dfrac{1}{3!}$.

(c) $\operatorname{Res}_{z=-1} f(z) = -\dfrac{17}{54e}$, $\operatorname{Res}_{z=2} f(z) = \dfrac{e^2}{27}$, $\operatorname{Res}_{z=\infty} f(z) = \dfrac{17}{54e} - \dfrac{e^2}{27}$.

(d) $\operatorname{Res}_{z=2} f(z) = 1/2$, $\operatorname{Res}_{z=\infty} f(z) = -1/2$. Here we take advantage of the basic theorem on residues. Expansion of the function $(z-2)\exp(1/(z-2))$ in powers of z is a rather cumbersome problem.

667. (a) $-\dfrac{3}{2}\pi i$; (b) πi; (c) $-\pi i/\sqrt{2}$; (d) πi.

669. (a) $\pi/\sqrt{2}$; (b) $\pi/ab\,(a+b)$; (c) $2\pi/3$;

(d) $\dfrac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots 2n}\pi = \dfrac{\pi(2n)!}{2^{2n}(n!)^2}$.

670. (a) $\pi e^{-a}/2$; (b) 0. *Hint.* Consider the function

$$f(z) = \frac{z^2}{(z^2+1)^2} e^{tz}.$$

671. $I = \dfrac{\pi}{2a^2}(1 - e^{-a})$.

Chapter 12.

673. (a) $\dfrac{1}{p-2}$; (b) $\dfrac{3}{p^2+9}$. **674.** No, since $\varphi(p)$ is a periodic function.

675. (a) $\dfrac{1}{p^2} + \dfrac{2}{p}$; (b) $2 \cdot \dfrac{3}{p^2+9} + \dfrac{1}{p+2}$.

676. No, since the growth of the function $\exp(t^2)$ is higher than that of any exponential function $\exp(s_0 t)$ (for any constant s_0) as $t \to \infty$.

677. $\dfrac{p(p^2+m^2+n^2)}{(p^2+m^2+n^2)^2 - 4m^2n^2}$.

678. (a) $F(p)=\dfrac{p^2+18}{p(p^2+36)}$; (b) $F(p)=\dfrac{p^4+16p^2+24}{p(p^2+4)(p^2+16)}$. *Hint.* $f(t)=$
$=\cos^4 t$, $f(0)=1$, $F(p)\fallingdotseq f(t)$;

$$f'(t)=-4\cos^3 t\sin t=-2\cos^2 t\sin 2t$$
$$=-(1+\cos 2t)\sin 2t=-\sin 2t-\frac{1}{2}\sin 4t;$$
$$f'(t)\fallingdotseq-\frac{2}{p^2+4}-\frac{1}{2}\,\frac{4}{p^2+16}.$$

679. (a) $\dfrac{2p^3-6p}{(p^2+1)^3}\left(\left(\dfrac{p}{p^2+1}\right)''\fallingdotseq t^2\cos t\right)$; (b) $\dfrac{6p}{(p^2-9)^2}$.

680. (a) $\dfrac{6}{(p^2-9)^2}$; (b) $\dfrac{2p^2-6}{(p^2+1)^3}$ (see Problem 679).

681. (a) $\ln\dfrac{p}{p-1}$; (b) $\dfrac{1}{2}\ln\dfrac{\sqrt{p^2+4}}{p}$; (c) $\dfrac{1}{2}\ln\dfrac{p+1}{p-1}$.

682. (a) $\sin t\fallingdotseq\dfrac{1}{p^2+1}$, $e^{3t}\sin t\fallingdotseq\dfrac{1}{(p-3)^2+1}$;

(b) $\dfrac{2(p-1)^2-6(p-1)}{[(p-1)^2+1]^3}=\dfrac{2(p-1)(p-4)}{(p^2-2p+2)^3}$ (Problem 679).

683. (a) For the function $f(t)=\sin t\cdot\sigma_0(t)$ we have $f(t)\fallingdotseq\dfrac{1}{p^2+1}$, therefore
by the shift theorem, $\sin(t-b)\,\sigma_0(t-b)\fallingdotseq e^{-pb}\dfrac{1}{p^2+1}$; (b) $\dfrac{e^{-2t}}{p-1}$.

684. (a) $\dfrac{1-e^{-p}}{p^2(1+e^{-p})}$; (b) $\dfrac{1}{p^2+1}\,\dfrac{1+e^{-\pi p}}{1-e^{-\pi p}}$. *Hint.*

$$L[|\sin t|;p]=\sum_0^\infty\int_{k\pi}^{(k+1)\pi}e^{-pt}|\sin t|\,dt$$
$$=\sum_0^\infty\left[\int_{2k\pi}^{(2k+1)\pi}e^{-pt}\sin t\,dt-\int_{(2k+1)\pi}^{(2k+2)\pi}e^{-pt}\sin t\,dt\right]$$
$$=\sum_0^\infty\frac{1}{p^2+1}\left[e^{-(2k+1)p\pi}+e^{-2kp\pi}+e^{-2(k+1)p\pi}+e^{-p(2k+1)\pi}\right]$$
$$=\frac{1}{p^2+1}\left[\sum_0^\infty e^{-pk\pi}+\sum_1^\infty e^{-pk\pi}\right].$$

685. (a) $\dfrac{1}{p-1}\cdot\dfrac{1}{p^2+1}$; (b) $\dfrac{2}{p^3}\cdot\dfrac{p}{p^2-1}=\dfrac{2}{p^2(p^2-1)}$.

686. $f(t)=\dfrac{1}{2!}\,\dfrac{t}{1!}-\dfrac{1}{4!}\,\dfrac{t^3}{3!}+\dfrac{1}{6!}\,\dfrac{t^5}{5!}-\ldots$

687. $f(t)=-1+\dfrac{1}{2}e^t+\dfrac{1}{2}\cos t-\dfrac{1}{2}\sin t$. *Hint.* Expand the rational function $F(p)$ in partial fractions and make use of the table of transforms.

688. (a) $f(t)=\dfrac{1}{2}(e^{-t}-e^{-3t})$; (b) $f(t)=t-\sin t$.

690. (a) $e^{-2t}\sin t$; (b) $\frac{t^2}{2}+2e^{-t}\sin t$.

691. (a) $x(t)=(t+1)e^{-t}$; (b) $x(t)=(t-1)\sin t$;

(c) $x(t)=\frac{3}{5}+\frac{2}{5}e^{-t}\cos 2t+\frac{1}{5}e^{-t}\sin 2t$.

692. (a) $x(t)=t\operatorname{arc\,tan} t-\frac{1}{2}\ln(1+t^2)$. *Hint.* First consider the problem $x''=1$, $x(0)=x'(0)=0$. Its solution: $x_1(t)=t^2/2$. By Duhamel's formula,

$$x(t)=\int_0^t \frac{1}{1+\tau^2}(t-\tau)\,d\tau,$$

(b) $y=(\sin x-x\cos x)/2$; (c) $y=e^{2x}+4\cos x-2\sin x-5$.

694. (a) $x(t)=e^t$, $y(t)=e^t$;

(b) $x(t)=\frac{3}{4}(1-t)-\frac{3}{4}\cos 2t+\frac{3}{8}\sin 2t$,

$y(t)=\frac{3}{4}(1-t)+\frac{1}{4}\cos 2t-\frac{1}{8}\sin 2t-\cos t$,

$z(t)=\cos t$;

(c) $y(x)=\frac{3}{4}e^{2x}+\frac{1}{4}e^{-2x}-e^x$, $z(x)=\frac{3}{4}e^{2x}-\frac{1}{12}e^{-2x}-\frac{2}{3}e^x$.

695. (a) $\frac{\pi}{2a}e^{-ax}$. *Hint.*

$$L[I(x);p]=\int_0^\infty\left(\int_0^\infty e^{-px}\cos xt\,dx\right)\frac{dt}{a^2+t^2}$$

$$=\int_0^\infty\frac{p}{p^2+t^2}\cdot\frac{dt}{a^2+t^2}=\frac{p}{a^2-p^2}\int_0^\infty\left[\frac{1}{p^2+t^2}-\frac{1}{a^2+t^2}\right]dt$$

$$=\frac{p}{a^2-p^2}\left[\frac{1}{p}\operatorname{arc\,tan}\frac{t}{p}-\frac{1}{a}\operatorname{arc\,tan}\frac{t}{a}\right]_0^\infty=\frac{\pi}{2}\cdot\frac{p}{a^2-p^2}\cdot\frac{a-p}{ap}=\frac{\pi}{2a}\cdot\frac{1}{a+p}.$$

Hence, $I(x)=\frac{\pi}{2a}\exp(-ax)$.

(b) $$L[I(x);p]=\int_0^\infty\left(\int_0^\infty e^{-px}\sin xt\,dx\right)\frac{\cos t}{t}\,dt$$

$$=\int_0^\infty\frac{t}{p^2+t^2}\cdot\frac{\cos t}{t}\,dt=\int_0^\infty\frac{\cos t\,dt}{p^2+t^2}=\frac{\pi}{2p}e^{-p}.$$

According to the table of transforms $I(x)=\frac{\pi}{2}\sigma_0(x-1)$, $x\neq 1$. If $x=1$, then $I(1)=\frac{1}{2}\int_0^\infty\frac{\sin 2t}{t}\,dt=\frac{1}{2}\int_0^\infty\frac{\sin u}{u}\,du=\frac{\pi}{4}$.

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